Resource cost in the classical simulation of a quantum preparation-measurement process

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Why is the standard interpretation of QM unsatisfactory?

- No description of the actual reality at the microscopic level

- Measurement problem

- exponential growth of resources in the definition of the state

Exponential growth

$$\psi(\vec{x}_1,\ldots,\vec{x}_N)$$

M lattice points for each coordinate

$$2M^{3N} \text{ real variables to define } \psi \Rightarrow$$

$$\text{Exponential growth of the information}$$

Classical analogy:
multi-particle probability distribution

$$\rho(\vec{x}_1, \vec{p}_1,\ldots,\vec{x}_N, \vec{p}_N)$$

quantum vs classical mechanics and exponential complexity

	Quantum	Classical
Ensemble description:	$\psi(x_1,..,x_N)$	$\rho(x_1, p_1,..,x_N, p_N)$
Single system description:	**?**	$\{x_1, p_1,..,x_N, p_N\}$

-The lacking of a single system description in QM implies that a exponentially growing number of resources is necessary to define a state

This does not occur in the classical single system description: the number of variables scales as the number of particles

Ontological theories of QM

An ontological theory provides a description of a single system by means of well-defined variables

$$\psi \text{->} \rho(X|\ \psi)$$

$X \rightarrow$ set of ontological variables (continuous and/or discrete).

Quantum state: complete information about the probability of any event.

Ontic state: information about the actual state of a single system

Inf. ontic state < Inf. quantum state?

examples of ontological theories:

- Beltrametti-Bugajski model
- Bohm-de Broglie mechanics

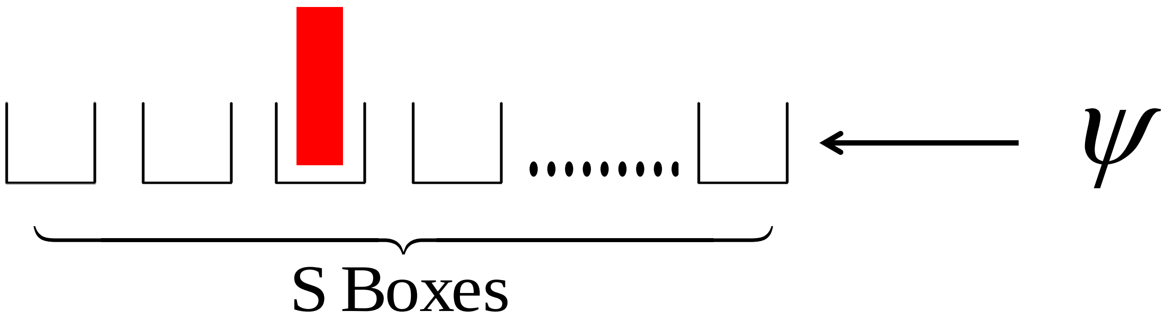
In **any** known exact ontological theory *the wave-function is promoted to the rank of a real field*-> The number of ontological resources grows exponentially!

Is this promotion necessary? Is it possible to reduce the number of required resources in order to define the state of a single system?

A. Montina, arXiv:quant-ph/0604155.

Why this question is not trivial?

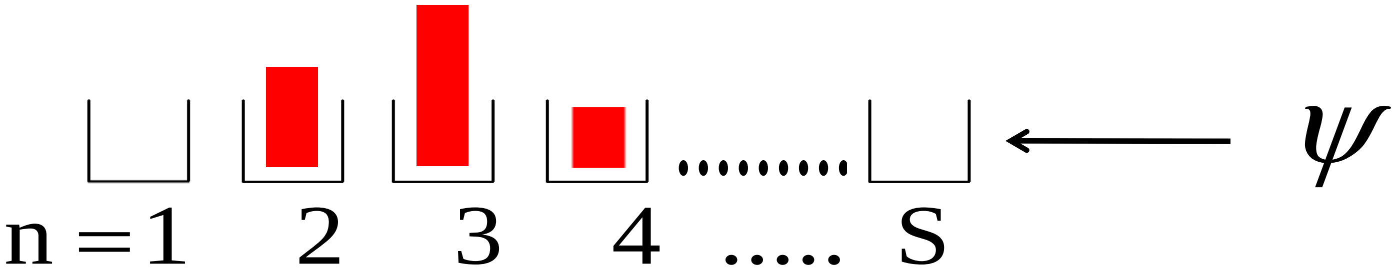
Example: the ontological space contains S ontic states



For each quantum state, only an ontic state is populated with probability = 1

Only S quantum states can be represented by the ontic space. **The ontic state set and the set of represented quantum states must have the same cardinality.**

Second example:



The probability distribution $\rho(n|\psi)$ is not delta-peaked

The number of quantum states one may represent is **in principle** much larger than S.

The dimensional reduction of the ontic space might be possible

A first constraint for the ontological space size:
L. Hardy's "*excess baggage theorem*".

The number of ontic states required to represent a finite dimensional Hilbert space can not be finite.

However, this result does not imply that the ontic space dimension can not be smaller than the Hilbert space dimension
(a 1D ontic space has infinite elements).

The dimensional reduction of the ontic space is not forbidden by Hardy's theorem

The set of ontic states is infinite, but is it countable or has the cardinality of continuum?

and, if the ontic space is continuous, what is its dimension?

In order to provide a more restrictive constraint, I used an additional hypothesis:

the dynamics has short memory (is Markovian).

A more restrictive theorem (Montina PRA, 2008)

In ANY ontological Markovian (deterministic or stochastic) theory of QM, the ontological variables contain a field ϕ whose space is isomorphic to the Hilbert space. The dynamics of this field is described by the Schrodinger equation:

$$i\hbar \frac{\partial \phi}{\partial t} = \hat{\mathrm{H}} \phi$$

Markovian process: it is a short memory process (deterministic or stochastic).

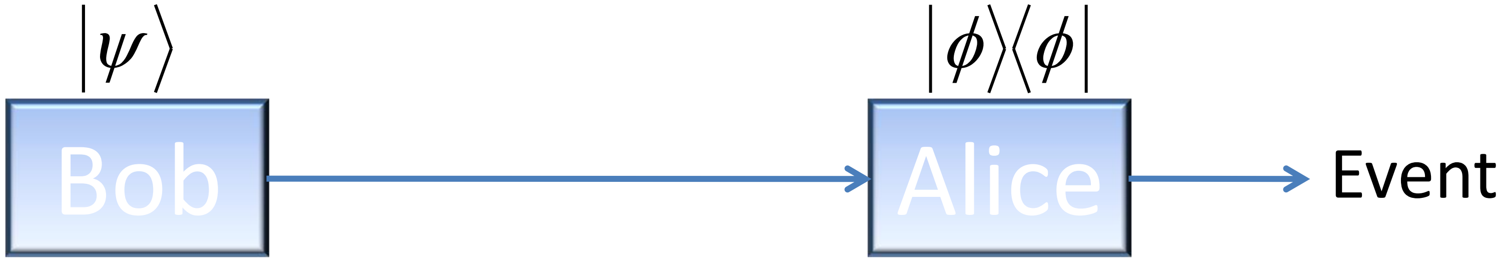
Consequence

The exponential growth of resources required to describe a system is not due to the ensemble description of the standard formulation, but it is intrinsic of ANY realistic Markovian theory of quantum mechanics.

Constructive result

We can reverse the theorem: in order to avoid the exponential growth, we could discard one of the theorem hypotheses, as the *Markovian hypothesis*.

Ontological model for state preparation-measurement processes



- Bob generates $|\psi>$ and sends to Alice a classical variable X with probability $\rho(X|\psi)$.

- Alice uses X to generate the event $|\phi><\phi|$ with probability $P(\phi|X)$.

- The probability of $|\phi><\phi|$ given $|\psi>$ is

$$\int P(\phi\,|\,X)\rho(X\,|\,\psi)dX = \left|\langle\phi|\psi\rangle\right|^2$$

Trivial scheme: Bob sends all the information about $|\psi>$.
Alice uses the Born rule to generate the event.

$$\rho(X\,|\,\psi) = \frac{1}{N^2}\delta(X-\psi)$$

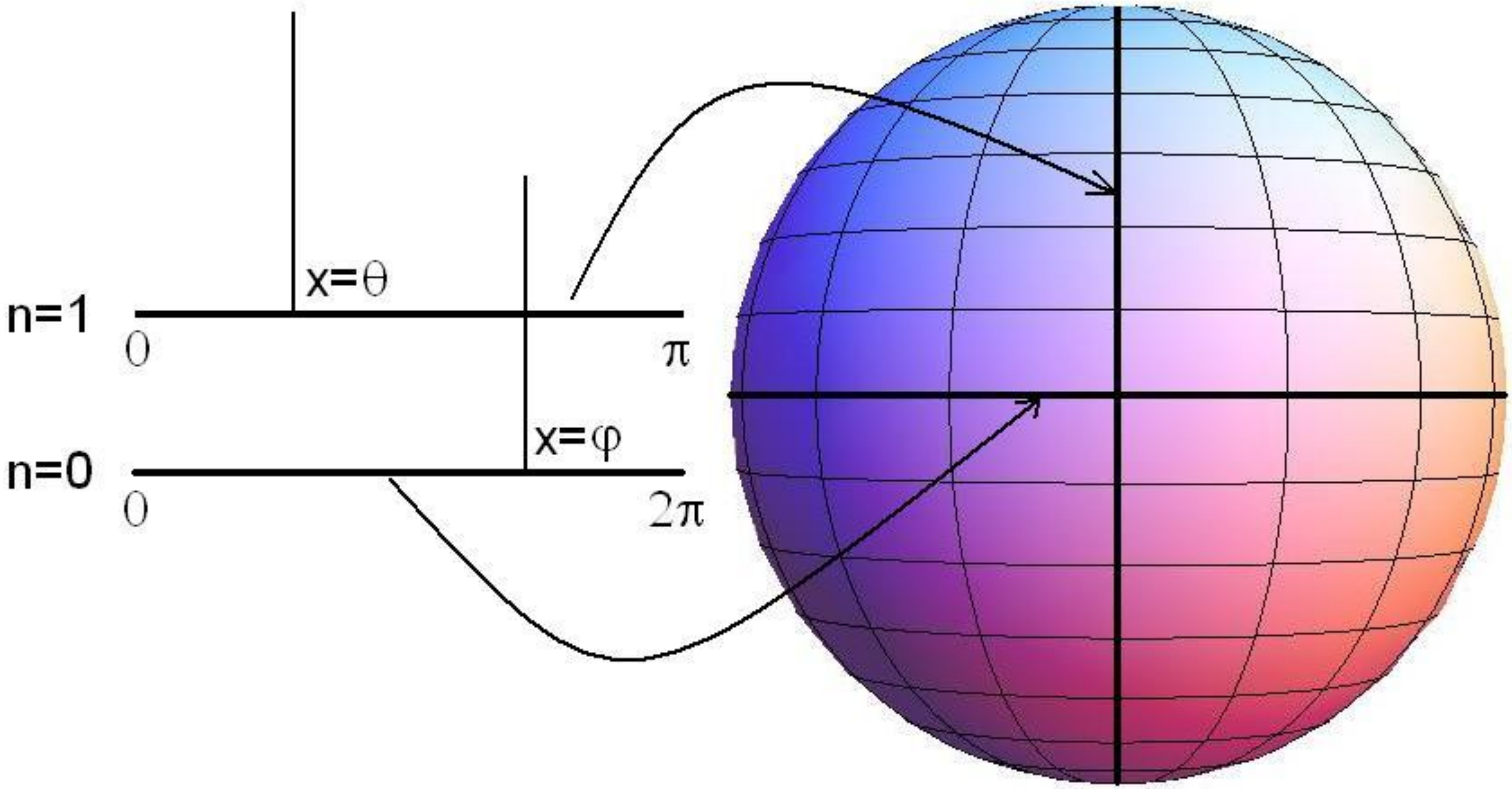
$$P(\phi\,|\,X) = \left|\langle\phi\,|\,X\rangle\right|^2$$

ontological model with ontic space shrinking for a qubit

-The quantum state manifold is **2 dimensional** (Bloch sphere), but the ontological space is **1 dimensional**! (ontic shrinking).
(the Kochen-Specker ontological space is 2 dimensional).

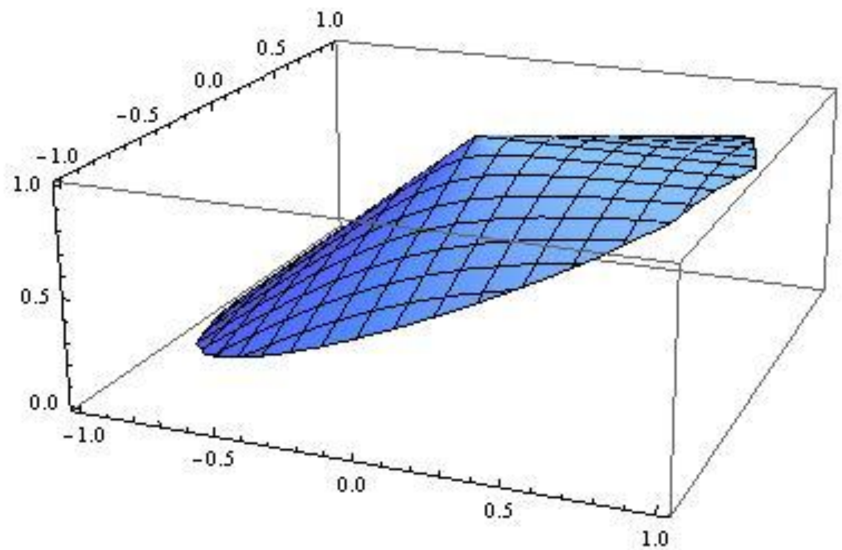
-The dynamics of the probability distribution is not described by a positive conditional probability (non-Markovian dynamics)

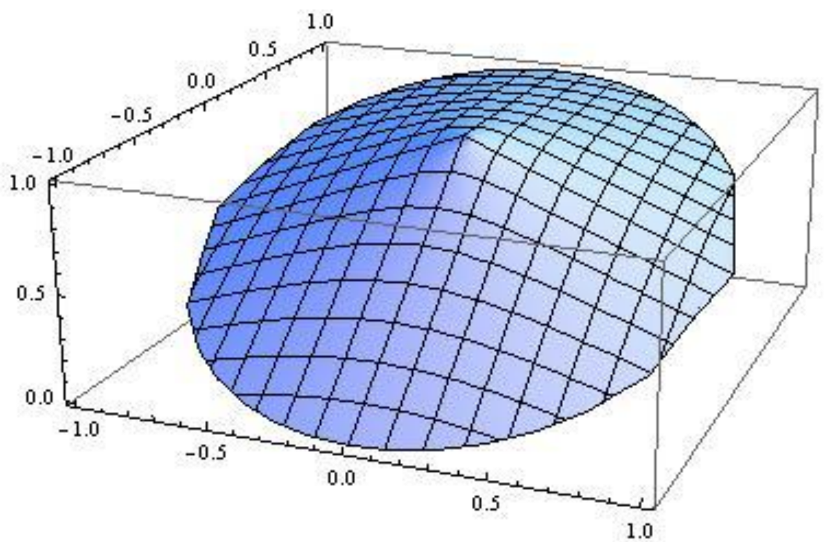
$$\rho(x, n|\vec{v}) = \sin\theta\delta_{n,0}\delta(x-\varphi) +$$
$$(1-\sin\theta)\delta_{n,1}\delta(x-\theta)$$

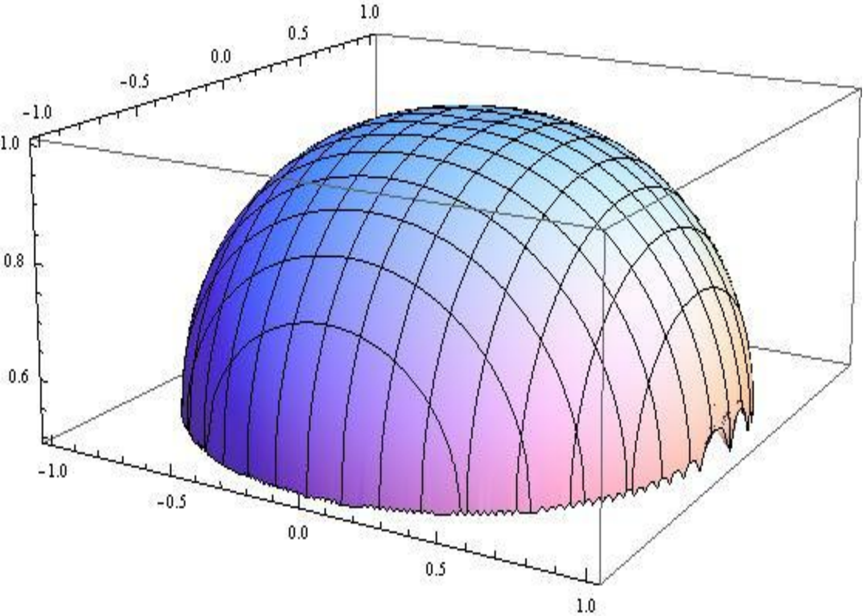


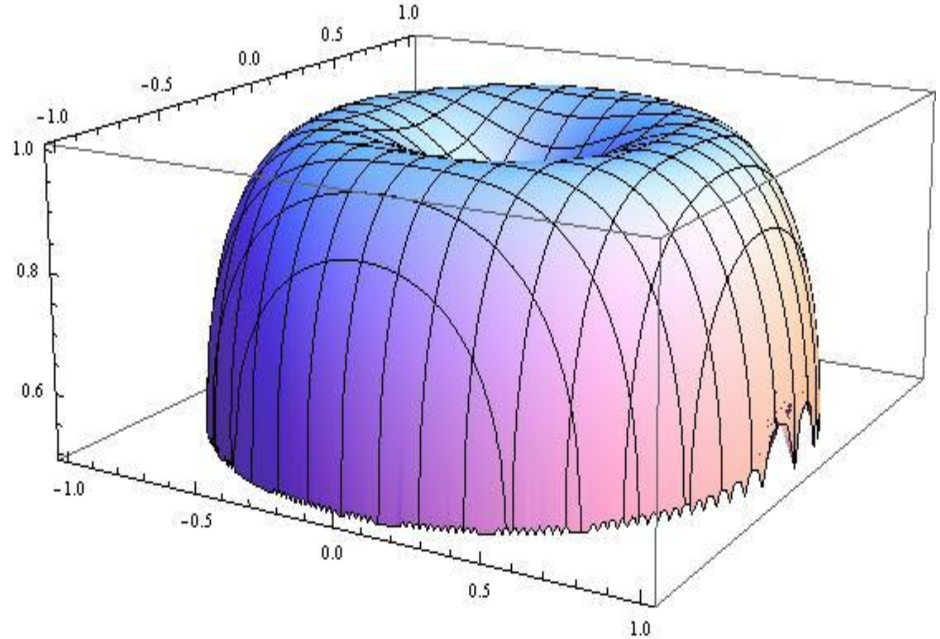
$$P(\vec{w}\,|\,x,0) = 1 + \frac{w_x \cos x + w_y \sin x - \sqrt{1 - w_z^2}}{2}$$

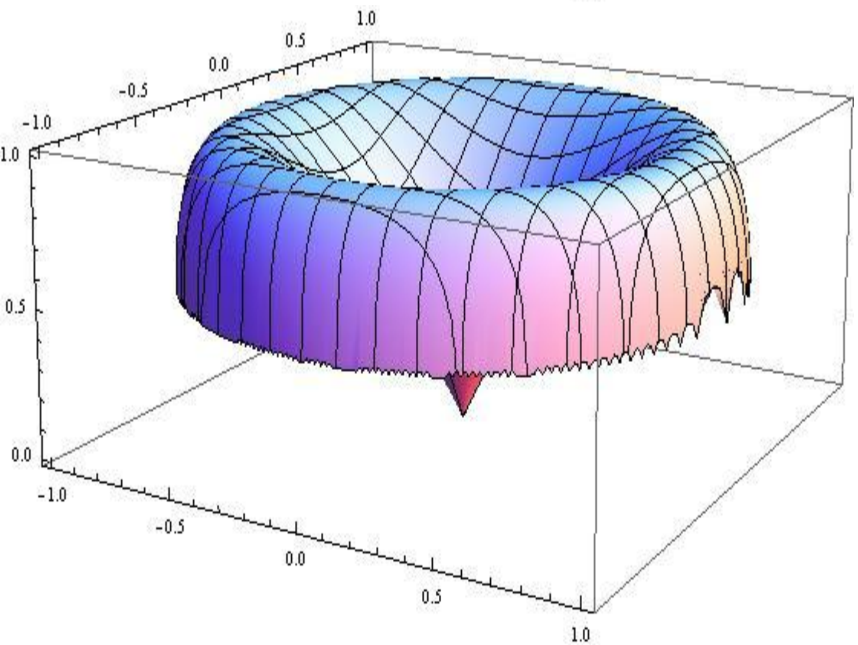
$$P(\vec{w}\,|\,x,1) = \frac{1 + (\sqrt{1 - w_z^2} - 2)\sin x + w_z \cos x}{2(1 - \sin x)}$$











$x<\mathrm{acos}(3/5)\cong 53.13$ degrees

- The model works for a cone of states on the Bloch sphere with symmetry axis z

Generalizing the model to any preparation state

$$\rho(x, n, m|\vec{v}) = \delta_{m, f(\vec{v})}(\dots)$$

Discrete index

Higher Hilbert space dimension

$$\psi \Rightarrow \rho(X, n, m | \psi) = R(n, m \mid X)\delta[X - \psi_n^* \psi_m]$$

where X is a complex number

$$\mathrm{P}(\phi | X, n, m) = 1 - \frac{1}{2R(n, m \mid X)} \left| X - \phi_n^* \phi_m \right|^2$$

$$P(\phi \mid X, n, m) > 0 \Rightarrow \left| X - \phi_n^* \phi_m \right|^2 < 2R(n, m \mid X)$$

Simple case:

$$R(n,m \mid X) = \frac{1}{N^2} \Rightarrow \left|\psi_k^* \psi_l - \phi_k^* \phi_l\right|^2 < \frac{2}{N^2} \Rightarrow \left|\psi_k - \phi_k\right|^2 \le \frac{2}{N^2}$$

Other choice:

$$R(k,l) = \frac{1}{4(N-1)^2}, \quad k,l \neq 0$$

$$R(0,k) = R(k,0) = \frac{1}{4(N-1)}, \quad k \neq 0$$

$$R(0,0) = \frac{1}{4}$$

$$P(\phi \mid X, n, m) > 0 \Rightarrow \left|\psi\right|^2 < \frac{1}{8(N-1)}, \ \left|\phi\right|^2 < \frac{1}{8(N-1)}$$

Monte Carlo Method

Ontological model:

$$\Delta\varepsilon = \frac{1}{\sqrt{N_R}}\left(\sigma_s^2 + \frac{1}{N_l}\right)^{1/2} \propto \frac{1}{\sqrt{I_{tot}}}, \quad \sigma_s \approx 0.1$$

Quantum state:

$$\Delta\varepsilon = \frac{1}{N_p}\frac{1}{(N-1)^{1/2}} \propto \frac{1}{\sqrt{I_{tot}}}$$

For M spins : $I_{tot} \approx 2^M$

Growing the region of positivity of the model

Perturbative corrections:

$$\rho(X,k,l\,|\,\psi)=\left(\frac{1}{N^2}+\varepsilon R(\psi,k,l)\right)\delta\left[X-\psi_k^*\psi_l-\varepsilon h(\psi,k,l)\right]$$

$$P(\phi\,|\,X,k,l)=1-\left[\frac{N^2}{2}+\varepsilon g(\phi,X,k,l)\right]\left|\phi_k^*\phi_l+\varepsilon h(\phi,k,l)-X\right|^2$$

$$\sum_{k,l}\int P(\phi\,|\,X,k,l)\rho(X,k,l\,|\,\psi)dX=\left|\langle\phi|\psi\rangle\right|^2$$

$$h(\psi,k,l)=\frac{1}{N^2}\sum_{i,j}\Big[(A_{k,l}(i,j)-A_{i,j}(k,l))\psi_i^*\psi_j+2\psi_k^*\psi_l J_{k,l}(i,j)|\psi_i|^2|\psi_j|^2\Big]$$
$$+\sum_{i,j}\Big[L_{k,l}(i,j)|\psi_i|^2|\psi_j|^2-2\psi_k^*\psi_l L_{i,j}(k,l)\psi_i^*\psi_j\Big]$$
$$+2\psi_k^*\psi_l\sum_{i,j}\Big[H_k(\psi,i,j)+H_l(\psi,i,j)-2\delta_{k,l}H_k(\psi,i,j)\Big]$$

$$g(\phi,X,k,l)=-\sum_{i,j}\Big[H_k(\phi,i,j)+H_l(\phi,i,j)+\phi_i^*\phi_j L_{i,j}(k,l)\Big]$$
$$+\sum_{i,j}|\phi_i|^2|\phi_j|^2 J_{k,l}(i,j)+2\beta_{k,l}(X)$$

$$R(\psi,k,l)=-\frac{2}{N^4}\sum_{i,j}\Big[H_k(\psi,i,j)+H_l(\psi,i,j)-L_{i,j}(k,l)\psi_i^*\psi_j\Big]$$
$$-\frac{2}{N^4}\sum_{i,j}J_{k,l}(i,j)|\psi_i|^2|\psi_j|^2-\frac{4}{N^4}\beta_{k,l}(\psi_k^*\psi_l)$$

Conclusion and perspectives

- Constraint on the ontological space dimension in Markov ontological theories.
- Economical ontological model of a state preparation-measurement process for a qubit
- Generalization to a N-dimensional Hilbert space, working for a subset of quantum states and measurement.
- Searching for an ontological model working for any state and measurement
- Issues on the dynamics (long memory dynamics or non-causal theory?)