On the phase-quantization problem in stochastic mechanics

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Stochastic mechanics - Background

- Formal analogies between QM and stat. mech. well-known.
- Ex: Schrödinger eq. and diffusion eq. related by Wick rotation.
- Suggestive of possibly deeper correspondence.

Q: Can QM be embedded within a classical theory of Brownian motion?

- **Stochastic mechanics (SM)**:
  (Fenyés, **Nelson**, Guerra, Yasue, Morato, Davidson, Nagasawa)

Nelson: I

- Stochastic differential eq. for **x**(t):

$$d\mathbf{x}(t) = \mathbf{b}(\mathbf{x},t)dt + d\mathbf{W}(t)$$

$$d\mathbf{x}(t) = \mathbf{b}_*(\mathbf{x},t)dt + d\mathbf{W}_*(t)$$

- Wiener noise:

$$E_t\left[d\mathbf{W}\right] = 0, \qquad E_t\left[d\mathbf{W}_i d\mathbf{W}_j\right] = 2\upsilon\delta_{ij}dt$$

- Diffusion coefficient:

$$\upsilon = \frac{\hbar}{2m}$$

Nelson: II

- Fokker-Planck (FP) eqs. for $\rho(\mathbf{x},\text{t})$:

$$\partial_t \rho = -\nabla \bullet \left(\mathbf{b}\rho\right) + \tfrac{\hbar}{2m} \nabla^2 \rho$$

$$\partial_t \rho = -\nabla \bullet \left(\mathbf{b}_* \rho\right) - \tfrac{\hbar}{2m} \nabla^2 \rho$$

- Subtract FP's to get 'osmotic velocity':

$$\mathbf{u} = \frac{\hbar}{2m} \frac{\nabla \rho}{\rho} = \frac{1}{2}\left[\mathbf{b} - \mathbf{b}_*\right]$$

- Assume osmotic potential field $R(\mathbf{x},t)$:

$$\frac{\nabla R}{m} = \frac{\hbar}{2m} \nabla \ln \rho \to \rho = e^{2R/\hbar}$$

Nelson: III

- Average FP's to get continuity eq.:

$$\partial_t \rho = -\nabla \bullet \left(\tfrac{1}{2} \left[\mathbf{b} + \mathbf{b}_* \right] \rho \right)$$

- Assume 'current velocity':

$$\mathbf{v} = \frac{\nabla S}{m} = \frac{1}{2} \left[\mathbf{b} + \mathbf{b}_* \right]$$

- Then $\mathbf{b} = \mathbf{v} + \mathbf{u}$ and $\mathbf{b}_* = \mathbf{v} - \mathbf{u}$.

Nelson: IV

- Stochastic derivatives:

$$D = \partial_t + \mathbf{b} \bullet \nabla + \tfrac{\hbar}{2m} \nabla^2 \qquad D_* = \partial_t + \mathbf{b}_* \bullet \nabla - \tfrac{\hbar}{2m} \nabla^2$$

$$D\mathbf{x} = \mathbf{b} \qquad D_*\mathbf{x} = \mathbf{b}$$

- T-symmetric mean acceleration:

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{1}{2}\left(D_* D + D D_*\right)\mathbf{x} = -\frac{\nabla V}{m}$$

$$\downarrow$$

$$\nabla\left(\partial_t S + \frac{(\nabla S)^2}{2m} + V + Q\right) = -\frac{\nabla V}{m}$$

Nelson: V

- Modified (quantum) Hamilton-Jacobi eq.:

$$-\partial_t S = \frac{(\nabla S)^2}{2m} + V + Q, \quad Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = \frac{1}{2}mu^2 - m\frac{\hbar}{2m}\nabla \bullet \mathbf{u}$$

- Hamilton-Jacobi-Madelung (HJM) eqs.:

$$-\partial_t S = \frac{(\nabla S)^2}{2m} + V + Q, \quad \& \quad \partial_t \rho = -\nabla \bullet \left(\frac{\nabla S}{m}\rho\right)$$

$$\Downarrow$$

- SE:

$$i\hbar\partial_t \psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi, \qquad \psi = \sqrt{\rho}e^{iS/\hbar}$$

Contrast SM with deBB

- SM ψ is derived and epistemic, *not* fundamental and ontic.
- Guiding eq. not exact but mean velocity: $\mathbf{v} = \dfrac{\nabla S}{m} = \dfrac{1}{2}\left[\mathbf{b} + \mathbf{b}_*\right]$
- S and ρ both required for particle dynamics (e.g. $\mathbf{b} = \mathbf{v} + \mathbf{u}$).
- Newton's 2nd law required for particle dynamics (HJM eqs.).
- Probability density ρ *not* a physical force field, even though used in osmotic velocity and quantum potential definition.
- SM undercuts Deustch/Wallace/Brown claim of "many-worlds in denial", even in quantum equilibrium case.

The phase-quantization problem: I

- AKA “Wallstrom’s criticism”.
- But first recognized by Takabayasi (’52).
- SE of QM and HJM not equivalent unless impose “ad-hoc” Bohr-Sommerfeld-Wilson (BSW) quantization condition:

$$\oint_L \nabla S \bullet d\mathbf{l} = nh \Leftrightarrow \oint_L d\phi_{QM} = 2\pi n$$

- Nelson relies on assumed equivalence between HJM and SE.
- But Nelson did not require BSW condition on his *S*.

The phase-quantization problem: II

- Wallstrom (‘88, ‘94):

  1. If S single-valued, ψ single-valued -
     ➔ excludes ψ’s with quantized angular momentum factors like exp[imφ].

  2. If S arbitrarily multi-valued, ψ multi-valued -
     ➔ have ψ’s with non-quantized angular momentum (e.g. 2-d central potential).

  3. True in simply-connected configuration spaces.

  4. True in 2-d or higher.

  5. True of all formulations of SM.

The phase-quantization problem: III

- Wallstrom ('94):

  "In the context of stochastic mechanics, it is very difficult to see how the circulation of the current velocity might be quantized in a natural way … There seems to be nothing in the particle-oriented world of stochastic mechanics which can lead to what is effectively a condition on the 'wave function'."

Proposed solutions: I

- **Carlen & Loffredo (‘89):**
  - Introduce SM analog of BSW condition on multiply-connected S^1.
  - Relate in natural way to topological properties of S^1.
  - Then QM and SM equivalent on multiply-connected S^1.

- **Wallstrom (‘94):**
  - Interpret HJM as equations for compressible fluid.
  - Then BSW quantization is initial condition conserved in time via Kelvin’s circulation theorem.

- **Smolin (‘06):**
  - Considers QM and SM on S^1.
  - Asserts that multi-valued and discontinuous ψ obtained from SM on S^1 is in $L^2(S^1)$ and thus solution of SE.
  - Claims no reason why ψ can’t be multi-valued.
  - Suggests argument for S^1 generalizes to higher dimensions.

Proposed solutions: II

- **~~Carlen & Loffredo ('89):~~** Need BSW condition in simply-connected space of 2-d or higher.

- **~~Wallstrom ('94):~~** No known physical justification for assuming BSW condition as privileged initial condition.

- **~~Smolin ('06):~~**
  - Example on S^1 artificial and trivial.
  - Wallstrom problems arise in simply-connected spaces of 2-d or higher.
  - Physical ψ's must be single-valued or else QEV of KE diverges at nodes of multi-valued ψ (Valentini).
  - Multi-valued ψ implies non-quantized energy and angular momentum (contradicts experimental facts).

Classical *Zitterbewegung* (ZBW): I

- Bohm ('57): Assume particle in rest frame oscillates with constant frequency ω_0. Then

$$\delta\phi_0(t) = \omega_0 \delta t_0$$

- Lorentz transform:

$$\delta\phi_0(\mathbf{x},t) = \omega_0 \gamma\left(\delta t - \mathbf{v}\bullet\delta\mathbf{x}/c^2\right)$$
$$= \frac{\omega_0}{mc^2}\left(\gamma mc^2 \delta t - \gamma m \mathbf{v}\bullet\delta\mathbf{x}\right)$$
$$= \frac{\omega_0}{mc^2}\left(E\delta t - \mathbf{p}\bullet\delta\mathbf{x}\right)$$

- If $m = m_e$ and $(\omega_0/m_e c^2) = 1/\hbar$ then $\omega_0 = \omega_c$ (*Zitterbewegung*).

Classical *Zitterbewegung*: II

- ➔ $$\delta S_0(\mathbf{x},t) = E\delta t - \mathbf{p}\bullet\delta\mathbf{x}$$

- Since ϕ continuous function of **x** and *t*, and oscillation is simply harmonic,

$$\oint_L \delta\phi_0(\mathbf{x},t) = 2\pi n$$

$$\downarrow$$

$$\oint_L \delta S_0(\mathbf{x},t) = nh$$

which is BSW condition for time held fixed.

Classical *Zitterbewegung*: III

- In NR limit $v << c$,

$$S(\mathbf{x},t) = -S_0(\mathbf{x},t) = m\mathbf{v}\bullet\mathbf{x} - \left[mc^2 + p^2/2m\right], \qquad \mathbf{v} = \frac{\nabla S}{m} = \frac{\hbar\mathbf{k}}{m}$$

- S satisfies classical HJ eq. (neg. mc^2) :

$$-\partial_t S = \frac{(\nabla S)^2}{2m}$$

- If ZBW particle position not known, have $\rho(\mathbf{x},t)$ over fictitious Liouville ensemble of identical ZBW particles and

$$\partial_t \rho = -\nabla \bullet \left(\frac{\nabla S}{m}\rho\right)$$

- S now function on spacetime since also function over fictitious ensemble.

- L in BSW condition now over momentum field for fictitious ensemble.

Digression: De Broglie

- Used equivalent argument ('23) involving "phase waves".
- Phase waves carried no energy-momentum, thus were non-physical or "non-material".
- De Broglie speculated phase waves may be resonant EM waves.
- Phase waves were precursor to pilot waves.
- But phase waves *not necessary* for BSW quantization argument.

Incorporating stochastic mechanics: I

- Assume Nelson's noise field interacts with ZBW particle.
- In *instantaneous rest frame* defined by Nelson's noise field, ZBW particle undergoes Brownian motion with no drift:

$$d\mathbf{x} = d\mathbf{W}$$

- Define mean rest frame as $E_t[d\mathbf{x}] = E_t[d\mathbf{W}] = 0$.
- Then mean ZBW phase is constant in space: $\underline{S}(t) = mc^2 t$.
- Lorentz transform to mean fixed frame with mean velocity $\underline{\mathbf{v}}$:

$$\underline{S}(\mathbf{x},t) = m\underline{\mathbf{v}} \bullet \mathbf{x} - \left[mc^2 + \underline{p}^2 / 2m \right], \qquad \underline{\mathbf{v}} = \frac{\nabla \underline{S}}{m}, \qquad \oint_L \delta \underline{S}(\mathbf{x},t) = nh$$

$$-\partial_t \underline{S} = \frac{(\nabla \underline{S})^2}{2m}, \qquad \partial_t \rho = -\nabla \bullet \left(\frac{\nabla \underline{S}}{m} \rho \right)$$

Incorporating stochastic mechanics: II

- Thus, in instantaneous fixed frame,

$$d\mathbf{x} = \underline{\mathbf{v}}dt + d\mathbf{W}$$

- Now assume in instantaneous fixed frame some $R(\mathbf{x},t)$ imparts osmotic velocity to ZBW particle (in addition to $\underline{\mathbf{v}}$) where

$$\mathbf{u} = \frac{\nabla R}{m} = \frac{\hbar}{2m}\nabla \ln\rho \rightarrow \rho = e^{2R/\hbar}$$

- Then

$$d\mathbf{x} = \left(\underline{\mathbf{v}} + \mathbf{u}\right)dt + d\mathbf{W}, \qquad \underline{\mathbf{b}} = \underline{\mathbf{v}} + \mathbf{u}$$

$$d\mathbf{x} = \left(\underline{\mathbf{v}} - \mathbf{u}\right)dt + d\mathbf{W}_*, \qquad \underline{\mathbf{b}}_* = \underline{\mathbf{v}} - \mathbf{u}$$

Incorporating stochastic mechanics: III

- Corresponding FP eq.'s:

$$\partial_t \rho = -\nabla \bullet \left(\underline{\mathbf{b}}\rho\right) + \tfrac{\hbar}{2m}\nabla^2 \rho, \qquad \partial_t \rho = -\nabla \bullet \left(\underline{\mathbf{b}}\rho\right) + \tfrac{\hbar}{2m}\nabla^2 \rho$$

- Subtract ➔

$$\mathbf{u} = \frac{\hbar}{2m}\frac{\nabla\rho}{\rho} = \frac{1}{2}\left[\mathbf{b} - \mathbf{b}_*\right]$$

- Add ➔

$$\partial_t \rho = -\nabla \bullet \left(\underline{\mathbf{v}}\rho\right)$$

$$\downarrow$$

$$\underline{\mathbf{v}} = \frac{\nabla \underline{S}}{m} = \frac{1}{2}\left[\underline{\mathbf{b}} + \underline{\mathbf{b}}_*\right]$$

Incorporating stochastic mechanics: IV

- Stochastic derivatives and mean acceleration:

$$D = \partial_t + \underline{\mathbf{b}} \bullet \nabla + \tfrac{\hbar}{2m} \nabla^2 \qquad\qquad D_* = \partial_t + \underline{\mathbf{b}}_* \bullet \nabla + \tfrac{\hbar}{2m} \nabla^2$$

$$D\mathbf{x} = \underline{\mathbf{b}} \qquad\qquad D_*\mathbf{x} = \underline{\mathbf{b}}$$

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{1}{2}\left(D_* D + D D_*\right)\mathbf{x} = -\frac{\nabla V}{m}$$

$$\downarrow$$

$$-\partial_t \underline{S} = \frac{\left(\nabla \underline{S}\right)^2}{2m} + V + Q$$

$$\underline{S}\left(\mathbf{x},t\right) = m\underline{\mathbf{v}} \bullet \mathbf{x} - \left[\underline{p}^2 / 2m + V + Q\right]$$

Incorporating stochastic mechanics: V

- $V \neq 0$ contributes additional phase term to $\underline{S}$.
- $Q \neq 0$ implies that osmotic potential R contributes to ZBW phase $\underline{S}$ (i.e. they are physically coupled).
- BSW quantization still follows since $\underline{S}$ still continuous function of $\mathbf{x}$ and t and mean oscillation still simply harmonic.
- In formal 'classical limit', $Q \rightarrow 0$, recover classical HJ equations with BSW quantization condition.

Open problems

- Extend ZBW model to obtain quantization condition for multi-periodic motions (e.g. relativistic Kepler problem).

- Extend ZBW model to relativistic field theory.

- Find physical mechanism for ZBW oscillations.

- Use quantum nonequilibrium in SM with ZBW to predict breakdown of BSW quantization.

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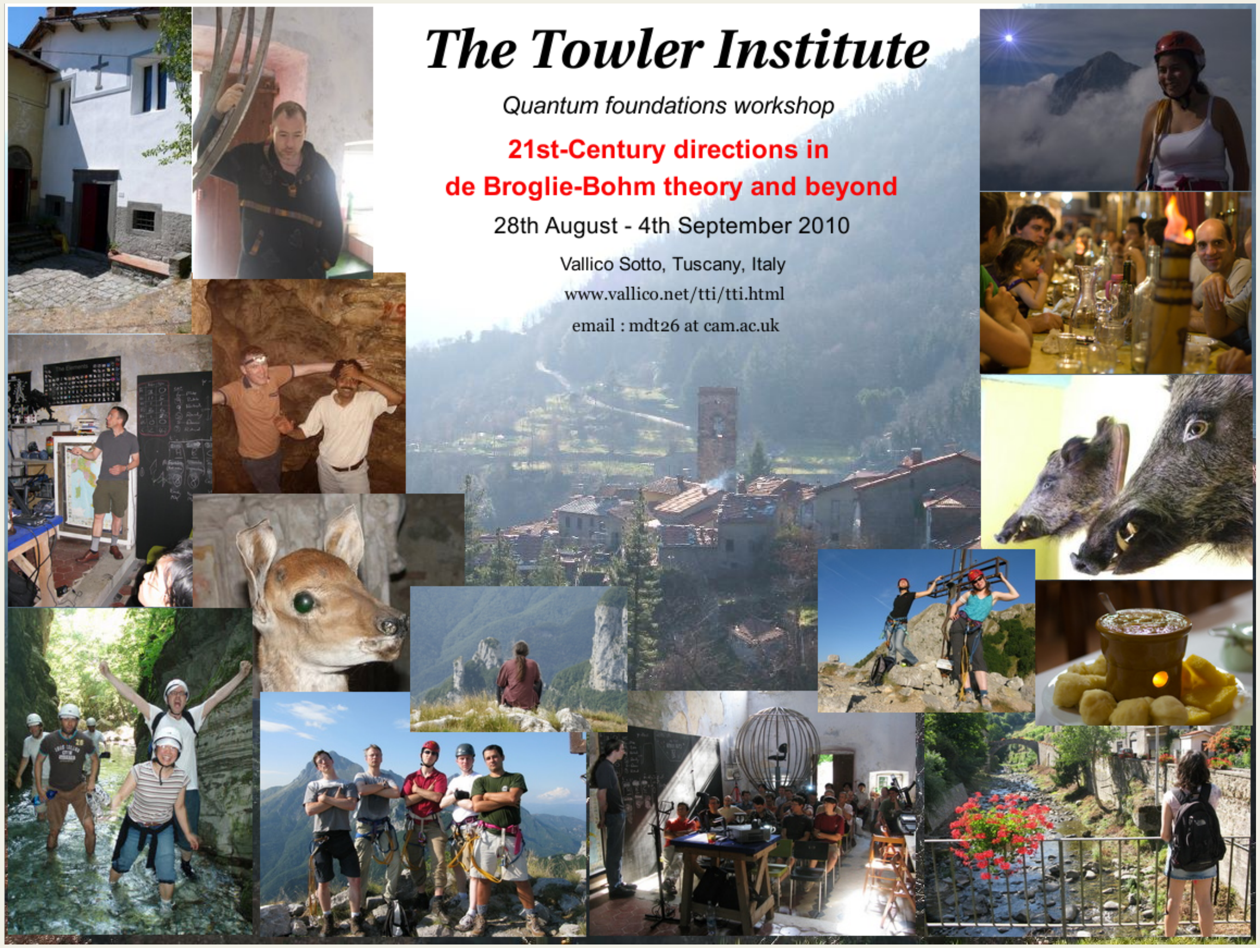

Ontology of SM with ZBW

① ZBW particle with oscillating trajectory (described by phase S).

② Stochastic noise field.

③ Osmotic potential field on configuration space.