Moyal and Clifford Algebras in the Bohm Approach.

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www.bbk.ac.uk/tpru.
The Bohm Story Unfolded.

Classical physics.

Things go on in space-time. \[ \text{dynamics} \rightarrow \text{phase space} \]

\[
\begin{array}{c}
\text{Particles or fields} \\
\downarrow \text{symplectic symmetry} \\
\text{x}
\end{array}
\]

Quantum physics.

abandon phase space \[ \rightarrow \text{Operators in Hilbert space} \]

\[
\begin{array}{c}
\text{Fields} \\
\downarrow \text{symplectic encoded in Heisenberg group} \\
\text{x}
\end{array}
\]

Primitive Bohm approach.

Looks like a return to particles in phase space.

Where has the symplectic symmetry gone? Its there still!

Can also construct and more!

\[
\begin{array}{c}
p = \nabla_x S \\
\downarrow \text{Particles} \\
x = -\nabla_x S
\end{array}
\]

[Brown, PhD thesis 2000]
[Brown and Hiley quant-ph/0005026]
Non-commutative Algebraic structure.

Process, activity

Holomovement

Implicate order.

Shadow phase space

$(X^\psi(x), P^\psi(x))$

Shadow phase space

$(X^{\phi(p)}, P^{\phi(p)})$

Shadow phase space

$(X^{\eta(x')}, P^{\eta(x')})$

Possible explicate orders.

[D. Bohm  Wholeness and the Implicate Order (1980)]
Can we lift the classical properties on to the covering spaces?

Dynamics, Symplectic Group.

Start with classical mechanics.

If $H$ is a function of $t$,

$$f_{t_0,t}^H \circ f_{t',t''}^H = f_{t,t''}^H$$

Hamiltonian groupoid

Motion is generated by Hamilton-Jacobi function $S(x, x', t, t')$

Lift this onto a covering space.

de Gosson has shown that $F_{t,t'}^H$ is Schrödinger for all Hamiltonians.

Key object:-

$$\rho(x, x', t) = \psi^*(x', t)\psi(x, t)$$

Non-local object

What is this object and how does it develop in time?

[de Gosson and Hiley, pre-print 2010]
The Two-point Density Matrix.

Start with

\[ \rho(x, x', t) = \psi^*(x', t) \psi(x, t) \]

Go to \( p \)-space

\[ \psi(x, t) = (2\pi)^{-1} \int \phi(p, t) e^{ipx} dp \]

\[ \rho(x, x', t) = (2\pi)^{-2} \int \int \phi^*(p', t) e^{ipx} \phi(p, t) e^{ix'p'} dp dp' \]

Use

\[ X = (x' + x)/2 \quad \eta = x' - x \quad P = (p' + p)/2 \quad \pi = p' - p \]

Then

\[ \rho(X, \eta, t) = (2\pi)^{-2} \int \int \phi^*(P - \pi/2, t) \phi(P + \pi/2, t) e^{iX\pi} d\pi e^{i\eta P} dP \]

Write as

\[ \rho(X, \eta, t) = (2\pi)^{-1} \int F(X, P, t) e^{i\eta P} dP \]

So that

\[ F(X, P, t) = (2\pi)^{-1} \int \phi^*(P - \pi/2, t) e^{iX\pi} \phi(P + \pi/2, t) d\pi \]

\[ = (2\pi)^{-1} \int \psi^*(X - \eta/2, t) e^{-i\eta P} \psi(X + \eta/2, t) d\eta \]

Try to use \( F(X, P, t) \) as a classical distribution function \( \Rightarrow \) negative probabilities

It is essentially a ‘density matrix’ in the \( (X, P) \) representation.

NB. It describes a ‘quantum blob’, not a classical particle.

Symplectic capacity
Quantum Phase space.

1. Change of representation $\Rightarrow$ return to phase space of functions?

   NB $X$ and $P$ are not coordinates of a simple particle.

   $[\hat{X}, \hat{P}] = 0$


2. Treat $F(X, P)$ as a quasi-probability density? Don’t!!

3. We can generate a new non-commutative algebra of functions with a new product.

   \[
   F(X, P) \ast G(X, P) = F(X, P) e^{i\hbar/2(\overline{\partial}_X \overline{\partial}_P - \overline{\partial}_P \overline{\partial}_X)} G(X, P)
   \]

   This product is non-commutative $F \ast G \neq G \ast F$

   But it is associative $F \ast (G \ast H) = (F \ast G) \ast F$

   We find that we can do quantum mechanics in the phase space without operators.

   No operators in Hilbert space!

   [Dubin, Hennings & Smith, Math. Aspects of Weyl Quantization, 2000]
Moyal * Multiplication is Matrix Multiplication.

Write in general

\[ A(X, P, t) = (2\pi)^{-1} \int \rho_{\alpha}(X - \eta/2, X + \eta/2) e^{-i\eta P} d\eta \]

write as

\[ A(X, P) = (2\pi)^{-1} \int \hat{A}(X, \eta) e^{-i\eta P} d\eta \]

Then

\[ A(X, P) \ast B(X, P) = C(X, P) \]

is equivalent to

\[ \hat{C}(X - \eta/2, X + \eta/2) = \int \hat{A}(X - \eta/2, y) \hat{B}(y, X + \eta/2) dy \]

Essentially matrix multiplication. \textbf{NB Non-local.}

For proof write

\[ A \ast B = \iint d\eta d\eta' \hat{A}(X - \eta/2, X + \eta/2) e^{-i\eta P} e^{i(\overline{\partial}_X \overline{\partial}_P - \overline{\partial}_P \overline{\partial}_X)} e^{-i\eta' P} \hat{B}(X - \eta'/2, X + \eta'/2) \]

[Cnockaert, Moyal’s Proc. Modave Summer School, 2005]
Properties of the Moyal *-Product?

**Moyal bracket** (commutator)

\[
\{A, B\}_{MB} = \frac{A \ast B - B \ast A}{i\hbar} = 2A(X, P) \sin \frac{\hbar}{2} \left[ \partial_X \partial_P - \partial_X \partial_P \right] B(X, P)
\]

**Baker bracket** (Jordan product or anti-commutator).

\[
\{A, B\}_{BB} = \frac{A \ast B + B \ast A}{2} = 2A(X, P) \cos \frac{\hbar}{2} \left[ \partial_X \partial_P - \partial_X \partial_P \right] B(X, P)
\]

Deform to obtain classical limit.

**Sine bracket** becomes **Poisson** bracket.

\[
\{A, B\}_{MB} = \{A, B\}_{PB} + O(\hbar^2)
\]

**Cosine bracket** becomes ordinary product

\[
\{A, B\}_{BB} = AB + O(\hbar^2)
\]

Quantum Dynamics.

Equation of motion for $\rho(x, x', t)$

$$-i \frac{\partial \rho}{\partial t} = H\rho - \rho H = \left[ \frac{1}{2m} \left( \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial x^2} \right) - V(x') + V(x) \right] \rho(x, x', t)$$

Changing variables $(x, x') \rightarrow (X, \eta)$ we find

$$-i \frac{\partial \rho}{\partial t} = \left[ \frac{1}{m} \frac{\partial}{\partial X} \frac{\partial}{\partial \eta} + V(X - \eta/2) - V(X + \eta/2) \right] \rho(X, \eta)$$

Write $V(X) = \int V_k e^{ikX} dk$ so that

$$V(X + \eta/2) - V(X - \eta/2) = \int V_k e^{ikX} (e^{ik\eta/2} - e^{-ik\eta/2}) dk$$

Use the Wigner-Moyal transformation $\rho(X, \eta, t) \rightarrow F(X, P, t)$

Finally

$$\frac{\partial F(X, P, t)}{\partial t} + \frac{P}{m} \frac{\partial F(X, P, t)}{\partial t} + i \int L(P, P') F(X, P') dP' = 0$$

Non-local transformation.
The Two Wigner-Moyal Equations.

Define two equations of motion

\[ H \ast F = i(2\pi)^{-1} \int e^{-i\eta p} \psi^*(x - \eta/2, t) \partial_t \psi(x + \eta/2, t) d\eta \]
\[ F \ast H = -i(2\pi)^{-1} \int e^{-i\eta p} \partial_t \psi^*(x - \eta/2, t) \psi(x + \eta/2, t) d\eta \]

[I have written for simplicity \( \eta \) for \( \hbar \eta \)]

Subtracting gives **Moyal bracket** equation

\[ i\hbar \partial_t F = \{H, F\}_{MB} \]

Classical Louville equation to \( O(\hbar) \)

Adding gives **Baker bracket** equation

\[ 2\{H, F\}_{BB} = i(2\pi)^{-1} \int e^{-i\eta p} [\psi^*(x - \eta/2, t) \partial_t \psi(x + \eta/2, t) - \partial_t \psi^*(x - \eta/2, t) \psi(x + \eta/2, t)] d\eta \]
\[ = i(2\pi)^{-1} \int e^{-\eta p} \psi^*(x - \eta/2, t) \overrightarrow{\partial_t} \psi(x + \eta/2) d\eta \]

Writing \( \psi = Re^{iS/\hbar} \) we obtain
Classical Limit.

We find

\[ \psi^* \partial_t \psi = \left[ \frac{\partial_t R(x + \hbar \eta / 2)}{R(x + \hbar \eta / 2)} - \frac{\partial_t R(x - \hbar \eta / 2)}{R(x - \hbar \eta / 2)} \right] + i \left[ \frac{\partial_t S(x + \hbar \eta / 2)}{S(x + \hbar \eta / 2)} - \frac{\partial_t S(x - \hbar \eta / 2)}{S(x - \hbar \eta / 2)} \right] \psi^* \psi \]

Expanding in powers of \( \hbar \)

\[ \{H, F\}_{BB} = -\frac{\partial S}{\partial t} F + O(\hbar^2) \]

which becomes

\[ \frac{\partial S}{\partial t} + H = 0 \]

Classical Hamilton-Jacobi

Deformation again gives classical mechanics.

Two key equations

\[ i\hbar \partial_t F = \{H, F\}_{MB} \]

Quantum Liouville

\[ -\frac{\partial S}{\partial t} + O(\hbar^2) = \{H, F\}_{BB} \]

Hamilton-Jacobi
Summary so far

1. We have constructed a non-commuting algebra in the covering structure of classical phase space.

2. This reproduces all the standard results of quantum mechanics.

3. We do not need operators in a Hilbert space.

4. This algebraic structure contains classical mechanics as a natural limit. No fundamental role for decoherence.

5. The structure is intrinsically non-local.

   CM uses point to point transformations in phase space.

   QM involve non-local transformations expressed through matrices.

   Basic unfolding and enfolding movements.
A New Order: the Implicate Order.

Enfolding-unfolding movement

Approximates Bohm trajectories?

Continuity of form not substance.

There are two types of order:-

Implicate order.

Explicate order.

[Bohm, Wholeness and the Implicate Order, 1980]
Evolution of Process in the Implicate Order.

Continuity of form

\[ eM_1 = M_2 e' \]

\[ e' = M_2^{-1} eM_1 \]

Evolution is an algebraic automorphism.

Assume:

\[ M_1 = M_2 = M \]

\[ M = \exp[iH\tau] \]

\[ \tau \text{ is the UNFOLDING PARAMETER.} \]

For small \( \tau \)

\[ e' = (1 - iH\tau)e(1 + iH\tau) \]

\[ i \frac{\partial e}{\partial \tau} = [H, e] \]

Think of \( e \) as the density operator \( \rho \). For pure states \( \rho \) is idempotent.

QUANTUM LIOUVILLE EQUATION OF MOTION.
Schrödinger Equation?

If we write formally $e = \psi \phi$ and place in

$$i \frac{de}{d\tau} = [H, e]$$

We find

$$i \frac{d\psi}{d\tau} \phi + i \psi \frac{d\phi}{d\tau} = (H\psi)\phi - \psi(\phi H)$$

Now split into two equations

$$i \frac{d\psi}{d\tau} = H\psi \quad \text{Schrödinger-like equation.}$$

$$-i \frac{d\phi}{d\tau} = \phi H \quad \text{Conjugate equation.}$$

Since $e \in \mathcal{A}$, what are $\psi$ and $\phi$?

[Baker, Phys. Rev. 6 (1958) 2198-2206.]
Minimal Ideals of the enfolding Algebra.

\[ \rho = |\psi\rangle\langle\phi| \Rightarrow \psi\langle\phi \Rightarrow \psi\epsilon\phi = \Psi_L\Psi_R \]

NB we use Dirac’s standard ket. \( \psi \in \mathcal{A} \)

Here \( \mathcal{E} \) is an idempotent \( \mathcal{E}^2 = \mathcal{E} \)

Symplectic spinors

\[ \Psi_L = \psi\epsilon \in \mathcal{A} \quad \text{Algebraic equivalent of a wave function} \]
\[ \Psi_R = \epsilon\phi \in \mathcal{A} \quad \text{Algebraic equivalent of conjugate wave function.} \]

Need two Schrödinger-like algebraic equations

\[ i \frac{\partial \Psi_L}{\partial t} = \overrightarrow{H} \Psi_L \quad -i \frac{\partial \Psi_R}{\partial t} = \Psi_R \overleftarrow{H} \]
The Two More Algebraic Equations.

Sum the two algebraic Schrödinger equations

\[ i \left[ (\overrightarrow{\partial}_t \Psi_L) \Psi_R + \Psi_L (\Psi_R \overrightarrow{\partial}_t) \right] = \left( \overrightarrow{H} \Psi_L \right) \Psi_R - \Psi_L \left( \Psi \overrightarrow{H} \right) \]

Write \( \rho = \Psi_L \Psi_R \) so that

\[ i \frac{\partial \rho}{\partial t} = [H, \rho]_- \]

Conservation of Probability

Subtract the two algebraic Schrödinger equations

\[ i \left[ (\overrightarrow{\partial}_t \Psi_L) \Psi_R - \Psi_L (\Psi_R \overrightarrow{\partial}_t) \right] = \left( \overrightarrow{H} \Psi_L \right) \Psi_R + \Psi_L \left( \Psi \overrightarrow{H} \right) \]

Polar decompose \( \Psi_L = Re^{iS} \epsilon \) and \( \Psi_R = \epsilon Re^{-iS} \)

\[ \rho \frac{\partial S}{\partial t} + \frac{1}{2} [H, \rho]_+ = 0 \]

Conservation of Energy.
Moyal and Quantum Algebraic Equations.

\[
\frac{\partial F}{\partial t} + [F, H]_{MB} = 0
\]
\[
i \frac{\partial \rho}{\partial t} + [\rho, H]_{-} = 0
\]
\[
2 \frac{\partial S}{\partial t} F + [F, H]_{BB} = 0
\]
\[
2 \frac{\partial S}{\partial t} \rho + [\rho, H]_{+} = 0
\]

Moyal algebra

Quantum algebra

Where is the quantum potential?
Project Quantum Algebraic Equations into a Representation.

Project into representation using $P_a = |a\rangle\langle a|$

$$i\frac{\partial P(a)}{\partial t} + \langle [\rho, H]_- \rangle_a = 0$$

$$2P(a)\frac{\partial S}{\partial t} + \langle [\rho, H]_+ \rangle_a = 0$$

Still no quantum potential

Choose $P_x = |x\rangle\langle x|$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{K\hat{x}^2}{2}$$

Quantum H-J equation.

Choose $P_p = |p\rangle\langle p|$

$$\frac{\partial S_x}{\partial t} + \frac{1}{2m}\left(\frac{\partial S_x}{\partial x}\right)^2 + \frac{Kx^2}{2} - \frac{1}{2mR_x}\left(\frac{\partial^2 R_x}{\partial x^2}\right) = 0$$

Conservation of probability

$$\frac{\partial S_p}{\partial t} + \frac{p^2}{2m} + \frac{K}{2}x_r^2 - \frac{K}{2R_p}\left(\frac{\partial^2 R_p}{\partial p^2}\right) = 0$$

Quantum potential

$$x_r = -\nabla_p S_p$$

The Overarching Structure

Non-commutative Algebraic structure.

Process, activity

Holomovement

Implicate order.

Possible explicate orders.

[D. Bohm  Wholeness and the Implicate Order (1980)]
Four Roads to Quantum Mechanics.

**Standard.**

Operators in Hilbert space.

**Generalized phase space.**

Uses ordinary functions in phase space with a non-commutative product.

- Moyal star product.
- Deformed Poisson algebra.

Advantage: Nice classical limit.

**Algebraic approach.**

Everything is done in the algebra.

Wave functions replaced by elements in the algebra.

Advantage: Uses Clifford algebra therefore includes Pauli and Dirac.

Can also Schrödinger exploiting $\mathbb{C} \cong C_{0,1}$

**de Broglie-Bohm.**

Contained in all of the above three.
Hierarchy of Clifford Algebras

- **Conformal**: \{1, e_0, e_1, e_2, e_3, e_4, e_5\}
- **Twistors**: \{\omega, \pi\}
- **Dirac**: \{1, e_0, e_1, e_2, e_3\}
  - \{1, \gamma_0, \gamma_1, \gamma_2, \gamma_3\}
- **Pauli**: \{1, e_1, e_2, e_3\}
  - \{1, \sigma_1, \sigma_2, \sigma_3\}
- **Schrödinger**: \{1, e_1\}
  - \{1, i\}

Quantum?
How does it work?

How do we specify the state of the system?

\[ \hat{\rho}(x, t) = \Phi_L(x, t)\Phi_R(x, t) = \phi_L(x, t)\epsilon\phi(x, t) = \phi_L(x, t)\epsilon\tilde{\phi}(x, t) \]

**Clifford density element**

How do we choose the idempotent?

Decided by the physics.

For Dirac \[ \epsilon = (1 + \gamma_0)/2 \] Picks a time frame

For Pauli \[ \epsilon = (1 + \sigma_3)/2 \] Picks direction of space

For Schrödinger \[ \epsilon = 1 \]

**NB we use Clifford algebras over the reals!**
Physical Content of Schrödinger.

\[ \phi_L = g_0 + eg_1 \quad \text{and} \quad \phi_R = \tilde{\phi}_L = g_0 - eg_1 \quad \text{e} \in C_{0,1} \]

\[ \rho = \Phi_L \tilde{\Phi}_L = \phi_L \tilde{\phi}_L = g_0^2 + g_1^2. \]

Relation to wave function: Cliff → Hilbert space.

\[ \phi_L \Rightarrow \psi_i \]

Then

\[ g_0 = (\psi^* + \psi)/2 \quad g_1 = i(\psi^* - \psi)/2 \]

If we write \( \psi = Re^{iS} \) then

\[ g_0 = R \cos(S) \quad g_1 = R \sin(S) \]

Then

\[ \rho = g_0^2 + g_1^2 = R^2. \]

satisfies

\[ i \frac{\partial \rho}{\partial t} + [\rho, H]_- = 0 \]

MISSING information about the phase!
Pauli Particle continued.

\[ \phi_L = g_0 + g_1 e_{23} + g_2 e_{13} + g_3 e_{12} \]
\[ \phi_L = RU \]
\[ \hat{\rho} = \Phi_L \tilde{\Phi}_L = \phi_L e \tilde{\phi}_L = R^2 U e \tilde{U} = R^2 (1 + U \sigma_3 \tilde{U}) / 2 \]
\[ \hat{\rho} = R^2 (1 + s \cdot \sigma) / 2 \]
\[ \rho s = \phi_L \sigma_3 \tilde{\phi}_L / 2 \]
\[ R^2 = \rho \]

It looks as if we have 4 real parameters to specify the state, \( \{\rho, s_1, s_2, s_3\} \)

But \( s^2 = 1/4 \)

Something missing again!
Dirac Particle.

\[ \phi_L = a + b\gamma_{12} + c\gamma_{23} + d\gamma_{13} + f\gamma_{01} + g\gamma_{02} + h\gamma_{03} + n\gamma_5. \quad \gamma \in C_{1,3} \]

\[ \hat{\rho} = \Phi_L \tilde{\Phi}_L = \phi_L \epsilon \phi_L = \phi_L (1 + \gamma_0) \tilde{\phi}_L / 2 \]

This will give 8-real dimension spinor. We need 4 complex spinor.

We need a different but related idempotent.

\[ \hat{\rho} = \phi_L (1 + \gamma_0 + i\gamma_{12} + i\gamma_{012}) \tilde{\phi}_L / 2 \]

Bi-linear invariants.

Only 7 independent. Need 8 \( \therefore \) Still one missing!

NB we describe physical processes by physical properties.

Dirac Current.

\[ J = \phi_L \gamma_0 \tilde{\phi}_L \]

With \( \phi_L = a + b\gamma_{12} + c\gamma_{23} + d\gamma_{13} + f\gamma_{01} + g\gamma_{02} + h\gamma_{03} + n\gamma_5 \).

To show it is the usual current we need Cliff \( \rightarrow \) Hilbert space.

\[ \phi_L \Rightarrow \psi_i \]

\[ \psi_1 = a - ib; \quad \psi_2 = -d - ic; \quad \psi_3 = h - in \quad \psi_4 = f + ig \]

After some work

\[ J^0 = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \]

\[ J^1 = \psi_1 \psi_4^* + \psi_2 \psi_3^* + \psi_3 \psi_2^* + \psi_4 \psi_1^* \]

\[ J^2 = i[\psi_1 \psi_4^* - \psi_2 \psi_3^* + \psi_3 \psi_2^* - \psi_4 \psi_1^*] \]

\[ J^3 = \psi_1 \psi_3^* - \psi_2 \psi_4^* + \psi_3 \psi_1^* - \psi_4 \psi_2^* \]

Dirac current in the standard representation.
What is Missing?

Phase information? Energy-momentum?

In conventional terms

\[ 2i T^{\mu\nu} = \bar{\psi} \gamma^\mu (\partial^\nu \psi) - (\partial^\nu \bar{\psi}) \gamma^\mu \psi = \bar{\psi} \gamma^\mu \partial^\nu \psi \]

In Clifford terms

\[ 2i T^{\mu\nu} = tr[\gamma^\mu \phi_L \gamma_{012} \partial^\nu \phi_L] \]

Only non-vanishing term in trace is when \( \phi_L \gamma_{012} \partial^\nu \phi_L \) is a vector.

After some work we find

\[ \phi_L \gamma_{012} \partial^\nu \phi_L = A_\nu(x^\mu) \gamma_\sigma \]

where

\[ A_0^\nu = -(a \partial^\nu b + c \partial^\nu d + f \partial^\nu g + h \partial^\nu n) \]

\[ A_1^\nu = -(a \partial^\nu g + b \partial^\nu f + c \partial^\nu h + d \partial^\nu n) \]

\[ A_2^\nu = (a \partial^\nu f - b \partial^\nu g - c \partial^\nu n + d \partial^\nu h) \]

\[ A_3^\nu = (a \partial^\nu n - b \partial^\nu h + c \partial^\nu f - d \partial^\nu g) \]
Bohm Energy-Momentum Density Dirac.

Using \[ \psi_1 = a - ib; \quad \psi_2 = -d - ic; \quad \psi_3 = h - in \quad \psi_4 = f + ig \]

\[ T^{00} = i \sum_{j=1}^{4} (\psi_j^* \partial^0 \psi_j - \psi_j \partial^0 \psi_j^*) = -\sum R_j^2 \partial^0 S_j \]

This is just the Bohm energy density, \( \rho E_B \)

\[ T^{0k} = -i \sum_{j=1}^{4} (\psi_j^* \partial^k \psi_j - \psi_j \partial^k \psi_j^*) = \sum R_j^2 \nabla S_j \]

This is just the Bohm momentum density, \( \rho P_B^k \)

Why do we call these Bohm energy-momentum?
Bohm Energy-Momentum for Pauli and Schrödinger

Pauli.

\[ 2\rho P^\mu = -i(\phi_L \sigma_3 \partial^\mu \tilde{\phi}_L) = 2\rho D^\mu \sigma_{123} \]

Where

\[ D^\mu = -(\partial^\mu g_0)g_2 + (\partial^\mu g_1)g_2 - (\partial^\mu g_2)g_1 + (\partial^\mu g_3)g_0 \]

Schrödinger

\[ E_B = -\sum_{i=1}^{2} R_j^2 \partial_t S_j \quad P_B = \sum_{i=1}^{2} R_j^2 \nabla S_j \]

\[ \rho E_B = -e(\phi_L \hat{\partial}_t \tilde{\phi}_L) = -R^2 \partial_t S \quad E_B = -\partial_t S \]

\[ \rho P_B = -e(\phi_L \hat{\nabla} \tilde{\phi}_L) = R^2 \nabla S \quad P_B = \nabla S \]

[Bohm and Hiley The Undived Universe, 1993]
Translations and Time Derivatives.

Construct a Clifford bundle.

\[ A' = gAg^{-1} \]

\[ V' = RV \]

Double cover algebraic spinors

\[ \tilde{D} = e_i \tilde{\nabla}_{e_i} \]

Spin bundle with connection.

N.B. We need TWO derivatives in the bundle space \( \tilde{D} \) and \( \vec{D} \).

\[ \tilde{D} = e_\mu \tilde{\partial}_\mu \]

\[ \vec{D} = \vec{\partial}_\mu e_\mu \]

Therefore we need to use two time development equations.
Time Evolutions: Differences and Sums.

Two equations for time evolution

\[ i \partial_t \Phi_L = \vec{H} \Phi_L \quad \text{and} \quad -i \partial_t \Phi_R = \Phi_R \vec{H} \]

\[ \vec{H} = H(D, V, m) \quad \text{and} \quad \vec{H} = H(D, V, m) \]

Difference:-

\[ i[(\partial_t \Phi_L)\tilde{\Phi}_L + \Phi_L(\partial_t \tilde{\Phi}_L)] = (\vec{H} \Phi_L)\tilde{\Phi}_L - \Phi_L(\tilde{\Phi}_L \vec{H}) \]

We can rewrite this as \[ i \partial_t \hat{\rho} = [H, \hat{\rho}] \quad \text{Liouville equation.} \]

Conservation of probability

Sum:-

\[ i[(\partial_t \Phi_L)\tilde{\Phi}_L - \Phi_L(\partial_t \tilde{\Phi}_L)] = (\vec{H} \Phi_L)\tilde{\Phi}_L + \Phi_L(\tilde{\Phi}_L \vec{H}) \]

Conservation of energy
Schrödinger Quantum Hamilton-Jacobi Equation

The LHS is

\[ i[(\partial_t \Phi_L)\tilde{\Phi}_L - \Phi_L(\partial_t \tilde{\Phi}_L)] = i\phi_L \vec{\partial}_t \tilde{\Phi}_L = 2\rho E_B = -2\rho \partial_t S \]

\[ 2\rho \partial_t S = (\vec{H}\Phi_L)\tilde{\Phi}_L + \Phi_L(\tilde{\Phi}_L \vec{H}) \]

Quantum Hamilton-Jacobi

Since we have written \( \Phi_L = R \exp(eS) \) with \( \epsilon = 1 \),

using

\[ H = \frac{P^2}{2m} + V(x) \]

\[ \partial_t S + \frac{(\nabla S)^2}{2m} - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} + V = 0 \]

Conservation of energy.

Quantum Potential
Back to the Two Key Equations.

\[ i\partial_t (\Phi_L \Phi_L) = (\overrightarrow{H} \Phi_L) \Phi_L - \Phi_L (\overrightarrow{\Phi_L H}) \]

\[ i\Phi_L \overleftarrow{\partial_t} \overleftarrow{\Phi_L} = (\overrightarrow{H} \Phi_L) \overleftarrow{\Phi_L} + \Phi_L (\overrightarrow{\Phi_L H}) \]

Shortened forms.

\[ i\partial_t \hat{\rho} = [H, \hat{\rho}]_- \]

\[ i\Phi_L \overleftarrow{\partial_t} \overleftarrow{\Phi_L} = [H, \hat{\rho}]_+ \]

- Probability
- Spin
- Energy
- Always produces a quantum potential

Quantum Liouville
Quantum H-J
Conservation equations
The Pauli Quantum Liouville Equation.

\[ i\partial_t\hat{\rho} = [H, \hat{\rho}] \]

LHS becomes

\[ i\partial_t\hat{\rho} = i\partial_t[\phi_L\epsilon\phi_L] = i\partial_t[\rho + \phi_L\sigma_3\phi_L] = i\partial_t\rho + 2\partial_t(\rho S) \]

\[ 2\rho S = i\phi_L\sigma_3\phi_L \]

PseudoSalar \hspace{1cm} Bivector

Look at Pseudoscalar part.

\[ [H, \hat{\rho}]_{\text{pseudo}} = (H\phi_L)\sigma_3\phi_L - \phi_L\sigma_3(H\phi_L) \]

\[ 2m[H, \hat{\rho}]_{\text{pseudo}} = 2i\rho\left[4S \cdot (P \cdot W) - \nabla P\right] = -2i\rho\left[(\nabla \ln \rho)P + \nabla P\right] = -2i\nabla \cdot (\rho P) \]

\[ \partial_t \rho + \nabla \cdot (\rho P / m) = 0 \]

Conservation of probability equation
The Bivector part of the QLE.

\[ [H, \hat{\rho}]_{\text{bivector}} = (\hat{H}\phi_L)\tilde{\phi}_L - \phi_L(\tilde{\phi}_L\hat{H}) \]

Then

\[ m\partial_t (\rho S) = -[\nabla P \cdot S + S \wedge \nabla W + P \cdot W] \]

Again after some tedious work we find

\[ \left( \partial_t + \frac{P \cdot \nabla}{m} \right) S = \frac{1}{m} \left[ \nabla^2 S + (\nabla \ln \rho) \nabla S \right] \wedge S \]

Remembering \( S = is \) and \( A \wedge B = i(A \times B) \)

\[ \frac{ds}{dt} = \left( \partial_t + \frac{P \cdot \nabla}{m} \right) S = \frac{1}{m} s \times \nabla(\rho \nabla s) \]

Equation for spin time evolution.

The Quantum Torque

Spin trajectories and orientations.

Fig. 4. Trajectories and orientations $\theta$ associated with figs. 1 and 2.
The Quantum Hamilton-Jacobi Equation.

\[ i\Phi_L \overleftarrow{\partial_t} \tilde{\Phi}_L = [H, \hat{\rho}]_+ \]

Working the LHS.

Writing

\[ i\epsilon = \sigma_{123}(1 + \sigma_3)/2 = (\sigma_{12} + \sigma_{123})/2 \]

\[ \rho\Omega_t \cdot S + i\rho\Omega_t = [H, \hat{\rho}]_+ \]

\[ \Omega_t = 2(\partial_t U)\tilde{U} \]

\[ \phi_L = RU \]

\[ \Omega_t \cdot S + 2i\Omega_t \Rightarrow \frac{\partial S_{\text{phase}}}{\partial t} \]

The scalar part using Euler angles gives the same as BST, namely

\[ E(t) = \Omega_t \cdot S = \partial_t \psi + \cos \theta(\partial_t \phi) \]

[Bohm, Schiller, Tiomno, Nuovo Cim., 1, (1955) 48-66]
The Quantum Hamilton-Jacobi Equation.

**Working the RHS of** \( \rho \Omega_t \cdot S + i \rho \Omega_t = [H, \hat{\rho}]_+ \)

Scalar part of \([H, \hat{\rho}]_+\) is \((\overrightarrow{H} \Phi_L) \Phi_L + \Phi_L (\Phi_L \overrightarrow{H})\)

Bivector part of \([H, \hat{\rho}]_+\) is \((\overrightarrow{H} \Phi_L) \overrightarrow{\sigma}_3 \Phi_L + \Phi_L \sigma_3 (\Phi_L \overrightarrow{H})\)

After tedious but straight forward working

\[
2m[H, \hat{\rho}]_+^{\text{scalar}} = 2\rho \left[2(S \cdot \nabla W) + P^2 + W^2\right]
\]

where \(W = \rho^{-1} \nabla (\rho S)\)

This becomes

\[
\Omega_t \cdot S = \frac{P^2}{2m} + \frac{1}{2m} \left[2(\nabla W \cdot S) + W^2\right]
\]

**Quantum Hamilton-Jacobi**

**Quantum Potential**
The Quantum Hamilton-Jacobi Equation.

\[2mQ = 2(\nabla W \cdot S) + W^2 = \left[ S^2\left( 2\nabla \ln \rho + (\nabla \ln \rho)^2 \right) \right] + S \cdot \nabla^2 S\]

Putting this all together we get the QHL equation

\[\frac{1}{2} \left[ \partial_t \psi + \cos \theta (\partial_t \phi) \right] + \frac{P^2}{2m} + Q = 0\]

where

\[Q = -\frac{\nabla^2 R}{2mR} + \frac{1}{8m} \left[ (\nabla \theta)^2 + \sin^2 \theta (\nabla \phi)^2 \right]\]

This is exactly the equation obtained in the BST theory.

Dirac Energy-Momentum Conservation Equation.

Slight difference.

\[
(\partial_\mu \partial^\mu \Phi_L)\tilde{\Phi}_L + \Phi_L (\partial_\mu \partial^\mu \tilde{\Phi}_L) + 2m^2 \Phi_L \tilde{\Phi}_L = 0
\]

\[
\Phi_L (\partial_\mu \partial^\mu \tilde{\Phi}_L) - (\partial_\mu \partial^\mu \Phi_L)\tilde{\Phi}_L = 0
\]

In order to proceed we need to start with

\[
2\rho P^\mu = [(\partial^\mu \phi_L)\gamma_{012}\tilde{\phi}_L - \phi_L \gamma_{012}(\partial^\mu \tilde{\phi}_L)]
\]

and use

\[
2\rho J = \phi_L \gamma_{012}\tilde{\phi}_L \quad \text{and} \quad 2\rho W^\mu = -\partial^\mu (\phi_L \gamma_{012}\tilde{\phi}_L)
\]

we find

\[
P^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] + [J\partial_\mu P^\mu - \partial_\mu P^\mu J] - m^2 = 0
\]

Separate Clifford scalar and pseudo-scalar parts, we find

\[
P^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] - m^2 = 0
\]

c.f.

\[
p_\mu p^\mu - m^2 = 0
\]
Dirac Continued.

We have

\[ P^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] - m^2 = 0 \]

but

\[ 4\rho^2 P^2 = 4\rho^2 P_B^2 + \sum_{i=1}^{3} A_{i\nu} A^\nu_i = 4\rho^2 P_B^2 + 4\rho^2 \Pi^2 \]

Thus

\[ P_B^2 + \Pi^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] - m^2 = 0 \]

Compare with

\[ p_\mu p^\mu - m^2 = 0 \]

Find the quantum potential is

\[ Q_D = \Pi^2 + W^2 + [J\partial_\mu W^\mu + \partial_\mu W^\mu J] \]

Compare with quantum potential of Pauli

\[ Q_P = W_P^2 + [S(\nabla W_P) + (\nabla W_P)S] \]

[2\rho S = \phi_L \sigma_{12} \tilde{\phi}_L]

Quantum potential of Schrödinger

\[ Q_S = -\frac{1}{2m} \frac{\nabla^2 R}{R} \]
Dirac Spin Torque.

Go back to  \[ \Phi_L(\partial_\mu \partial^\mu \Phi_L) - (\partial_\mu \partial^\mu \Phi_L)\Phi_L = 0 \]

and get  \[ J \cdot \partial_\mu P^\mu - P \cdot W + J \wedge \partial_\mu W^\mu = 0 \]

with  \[ 2J \cdot \partial_\mu P^\mu = J\partial_\mu P^\mu + \partial_\mu P^\mu J \]

\[ 2P \cdot W = PW + WP \]

\[ 2J \wedge \partial_\mu W^\mu = J\partial_\mu W^\mu - \partial_\mu W^\mu J \]

since  \[ \rho(P \cdot W) = -(\partial^\mu \rho)(P_\mu \cdot J) - \rho(P_\mu \cdot \partial^\mu J) \]

we find  \[ \partial_\mu (\rho P^\mu) \cdot J + \rho(P_\mu \cdot \partial^\mu J) + \rho(J \wedge \partial_\mu W^\mu) = 0 \]

Since  \[ 2\partial_\mu (\rho P^\mu) = \partial_\mu (T^{\mu 0}) = 0 \]

\[ P_\mu \cdot \partial^\mu J + J \wedge \partial_\mu W^\mu = 0 \]

Quantum torque equation for Pauli is

\[ \left( \partial_t + \frac{P \cdot \nabla}{m} \right) S = \frac{2}{m} (\nabla W \wedge S) \]
Conclusions.

1. Do quantum mechanics entirely within the Clifford algebra. No need for wave functions!

2. All terms used are bilinear invariants, i.e. observable quantities. No wave functions

3. Use local energy-momentum density $T^{\mu 0}(x^\mu)$

4. The Bohm model follows immediately.

$$2 \rho P^\mu_B(x^\mu) = T^{\mu 0}(x^\mu)$$

No appeal to classical mechanics at all.

Yet the Clifford is about classical space-time
What does it all mean Physically?

Process space

Not

Classical space

but

Process space

$\eta_1$

$\eta_2$

Classical space $\oplus$ Classical space

Implicate order.

Non-commutative Algebraic structure.

Process:- The holomovement

Shadow manifold

Possible explicate orders.
References.


