Lorentz–Invariant Relaxation to Quantum Thermal Equilibrium

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Pilot–Wave Theory

- there IS a particle, with position $\mathbf{x}$ at time t
- the particle motion is piloted by the phase of the QM wavefunction $\psi[\mathbf{x}, \mathbf{t}]$
- the particle density relaxes in time to the Born density $|\psi|^2$
- the collapse of ψ at measurement is determined by the particle position

Outline (1)

spin-0 Schrödinger equation

- de Broglie velocity, kinematics
- Born density $|\psi|^2$, particle density ρ , defect g
- homogeneous & stationary turbulence
- asymptotic log–normality of Jacobian J
- coarse–graining, defect variance, relaxation time scale

Outline (2)

Spin-0 Stückelberg equation

- parametric relativistic QM, normalization, parameter ‘clocks’
- natural extension of analysis of nonrelativistic dispersion

Schrödinger equation

wavefunction $\quad \psi \in L^2(X^3 \subset \mathbb{R}^3; d^3x)$

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi[x_j,t] = \left\{-\left(\frac{\hbar^2}{2m}\right)\frac{\partial^2}{\partial x_k \partial x_k} + V[x_j,t]\right\}\psi[x_j,t]$$

de Broglie velocity

current identity

$$\frac{\partial}{\partial t}|\psi|^2 = -\frac{\partial}{\partial x_k}\left(v_k|\psi|^2\right)$$

$$v_k = \frac{1}{m}\frac{\partial}{\partial x_k}S \quad \text{where} \quad \psi = |\psi|\exp(\mathrm{i}S/\hbar)$$

Kinematics

Particle path $x_j = P_j(a_k, s; t)$

guided by pilot wave

$$\frac{D}{Dt} P_j(a_k, s; t) = v_j[P(a_k, s; t), t]$$

from $P_j(a_k, s; s) = a_j$

labels a_k, s

conversely $a_k = P_k(x_j, t; s)$

Jacobian

Jacobi matrix

$$J_{kj}(a_l, s; t) = \frac{\partial}{\partial a_k} P_j(a_l, s; t)$$

determinant

$$J = \det(J_{kj})$$

Pictures

Eulerian $[x_k, t]$ Lagrangian $(a_j, s; t)$

$$F(a_j, s; t) = F[P_k(a_j, s; t), t]$$

$$F[x_k, t] = F(P_j(x_k, t; s), s; t)$$

$$\frac{D}{Dt} F(a_j, s; t) = \left(\frac{\partial}{\partial t} + v_i[x_k, t] \frac{\partial}{\partial x_i} \right) F[x_k, t]$$

where $x_k = P_k(a_j, s; t)$

Kinematic identity

consider comoving volume or ‘fluid parcel’

$$\frac{D}{Dt}J = J\Delta \equiv J\frac{\partial v_k}{\partial x_k}$$

hence

$$\ln J(a_j, s; t) = \int_s^t \Delta(a_j, s; r)dr$$

Born density $|\psi|^2$

$$\frac{D}{Dt}\left(|\psi|^2 J\right) = 0$$

$$\int_{X^3} |\psi|^2[x_k, t] d^3x = \int_{X^3} |\psi|^2(a_j, s; t) J(a_j, s; t) d^3a = X^3$$

QM expectation $\quad < x_j >= X^{-3} \int_{X^3} x_j |\psi|^2[x_k, t] d^3x$

Particle density ρ

$$\frac{D}{Dt}(\rho J) = 0$$

$$\int_{X^3} \rho[x_k, t] d^3x = \int_{X^3} \rho(a_j, s; t) J(a_j, s; t) d^3a = X^3$$

deBB expectation $\quad < x_j >= X^{-3} \int_{X^3} x_j \rho[x_k, t] d^3x$

Defect $g = f - 1$

$$g = \frac{\rho - |\psi|^2}{|\psi|^2}$$

Constant on any path

$$\frac{Dg}{Dt} = 0$$

Unbiased for all t, s

$$< g >= X^{-3} \int_{X^3} g|\psi|^2 d^3x = 0$$

Homogeneous Turbulence (1)

$$\ln J(a_j, s; t) = \int_s^t \Delta(a_j, s; r) dr$$

How to take averages?

Δ is not result of context–free operator, in either picture

No requirement to take QM expectation with density $|\psi|^2$

Homogeneous Turbulence (2)

Jointly random

$\Delta(a_j, s; r),\ \Delta(a_j, s; r'),\ \Delta(a_j, s; r''), \ldots$

$$\overline{\ln J(a_j, s; t)} \equiv \int_s^t \overline{\Delta(a_j, s; r)} dr$$

Marginal first moment

$$\begin{aligned} \overline{\Delta(a_j, s; r)} &= X^{-3} \int_{X^3} \Delta(a_j, s; r) J(a_j, s; r) d^3 a \\ &= X^{-3} \int_{X^3} \Delta[z_k, r] d^3 z = 0 \end{aligned}$$

Homogeneous Turbulence (3)

Thus $\overline{\ln J(a_j, s; t)} = 0$ and

$$\begin{aligned}\overline{\ln^2 J(a_j, s; t)} &= 2\sigma^2 \int_0^{|t-s|} (|t-s| - r) C(r) dr \\ &\sim 2\sigma^2 |t-s| \mathcal{L}\end{aligned}$$

where

$$\sigma^2 C(r) = \overline{\Delta(u)\Delta(u \pm r)}, \quad \mathcal{L} = \int_0^\infty C(r) dr$$

Lagrangian integral time scale $\mathcal{L}$

Homogeneous Turbulence (4)

J is asymptotically Log–Normal

$$\ln J \sim \mathcal{N}\left(0, \sqrt{2\sigma^2|t-s|\mathcal{L}}\right)$$

as $|t-s|/\mathcal{L} \to \infty$

Finally

$$\overline{J(a_j, s; t)} \sim \exp(\sigma^2|t-s|\mathcal{L})$$

$J(t)$ asymptotically independent of $\Delta(r), \ s < r < t$

Coarse–Graining

Cubical cell, side l, partitioned into N^3 subcells of side l/N

$$\{g\}[x_k,t] \equiv l^{-3}\int_{l^3} g[x_k,t]d^3x$$
$$\approx N^{-3}\sum_{n_j} g[x_k + y_k(n_j),t]$$

QM expectation of $\{g\}$ vanishes:

$$<\{g\}>=0$$

QM Variance of Defect

$$< \{g\}^2 >= N^{-6} \sum_{n_p} \sum_{n_q} < g[b_j(n_p), s] g[b_k(n_q), s] >$$

where

$$b_j(n_p) = P_j(x_r + y_r(n_p), t; s)$$

Subcell expansion

$$\frac{l^3}{N^3} \to \frac{l^3}{N^3}\overline{J} \sim \frac{l^3}{N^3}\exp(\sigma^2|t-s|\mathcal{L})$$

as $|t-s|/\mathcal{L} \to \infty$

Recall $< g >= 0$, so (initial) defect g is one-signed in cubes of side W

Relaxation to QTE

White Noise for large $|t-s|/\mathcal{L}$

$$< g[b_j(n_p), s]g[b_k(n_q), s] > \sim < g^2 > \delta_{n_p,n_q}$$

Hence QTE

$$\{\rho[x_k, t]\} \sim |\psi|^2[x_k, t]\left(1 + \mathcal{O}(N^{-3/2} < g^2 >^{1/2})\right)$$

Time Scale $\mathcal{T}_N$

$$|t - s| \sim \mathcal{T}_N = \frac{3 \ln(NW/l)}{\sigma^2 \mathcal{L}}$$

$l =$ side of coarse–graining cell

$N^3 =$ number of subcells

$W =$ side of subdomains where $g[a_j, s]$ is one–signed

$\sigma^2 =$ variance of divergence $\Delta(a_j, s; t)$

$\mathcal{L} =$ Lagrangian decorrelation time scale for Δ

Which assumptions matter?

- these are assumptions about the *de Broglie velocity*
- the divergence of velocity must decorrelate along paths
- there must be a Lagrangian decorrelation time scale $\mathcal{L}$
- statistical nonstationarity is allowed subject to a Lindeberg condition

Relativistic QM

Stückelberg wavefunction $\psi \in L^2(X^4 \subset \mathbb{R}^4; d^4x)$

real parameter θ

$$\mathrm{i}\hbar \frac{\partial}{\partial\theta}\psi[x^\mu, \theta] = \left(\frac{1}{2m}\pi^\nu \pi_\nu + V[x^\mu, \theta] \right) \psi[x^\mu, \theta]$$

$$x^\mu = (ct, x, y, z), \quad \eta^{\mu\nu} = (-1, +1, +1, +1)$$

$$\pi^\mu = p^\mu - (q/c)A^\mu, \quad p^\mu = \frac{\hbar}{\mathrm{i}}\partial^\mu$$

Stückelberg Current

$$\frac{\partial}{\partial\theta}|\psi|^2 + \partial_\mu \mathtt{j}^\mu = 0$$

$$\psi = |\psi| \exp(\mathrm{i}S/\hbar)$$

$$\mathtt{j}^\mu = \frac{1}{m}\left(\partial^\mu S - \frac{q}{c}A^\mu\right)|\psi|^2$$

Admits tachyons, for which

$$\mathtt{j}^\mu \mathtt{j}_\mu > 0$$

What is parameter θ?

Proper time τ where $c^2 d\tau^2 = -dx^\mu dx_\mu$

$$c^2 \left(\frac{D\tau}{D\theta} \right)^2 = -v^\mu v_\mu$$

$$\psi = \exp[i(\epsilon\theta + k_\mu x^\mu)], \quad k^\mu k_\mu = -m^2 c^2/\hbar^2$$

$$\left(\frac{D\tau}{D\theta} \right)^2 = 1$$

Any on–shell free particle is a ‘θ – clock’

Space–Time Normalization

$$\int_{X^4} |\psi|^2[x^\mu, \theta] d^4x = X^4$$

$$< x^\mu > [\theta] = X^{-4} \int x^\mu |\psi|^2[x^\nu, \theta] d^4x$$

Invariant volume element d^4x

Marginal Born density

$$B_0[ct, \theta] = X^{-3} \int |\psi|^2[ct, x, y, z, \theta] d^3x$$

Stückelberg Equation

- negative *energy* states propagate backward in time
- Klein paradox is resolved (pair creation at potential step)
- *zitterbewegung* is interpreted (e.g., neutrino, Kaon oscillations)
- many particles, variable number
- coordinate time is an operator: interference, diffraction, entanglement
- spin–1/2: “g”=2 (Arshansky & Horwitz, 1982)

Stückelberg Pilot Wave

$$x_k \Longrightarrow x^\mu, \quad t \Longrightarrow \theta, \quad d^3x \Longrightarrow d^4x$$

Lorentz–Invariant Relaxation to QTE

$$\{\rho[x^\mu, \theta]\} \sim |\psi|^2[x^\mu, \theta]\left(1 + \mathcal{O}(N^{-2} < g^2 >^{1/2})\right)$$

$$|\theta - \phi| \sim \Theta_N = \frac{4\ln(NW/l)}{\sigma^2\Lambda}$$

Relaxation to QTE

- Piloting by Stückelberg wavefunction , normalized over space–time
- if ψ not a mass eigenstate, de Broglie velocity is a 'confused sea'
- expected expansion of invariant volume implies Lorentz–invariant relaxation to QTE