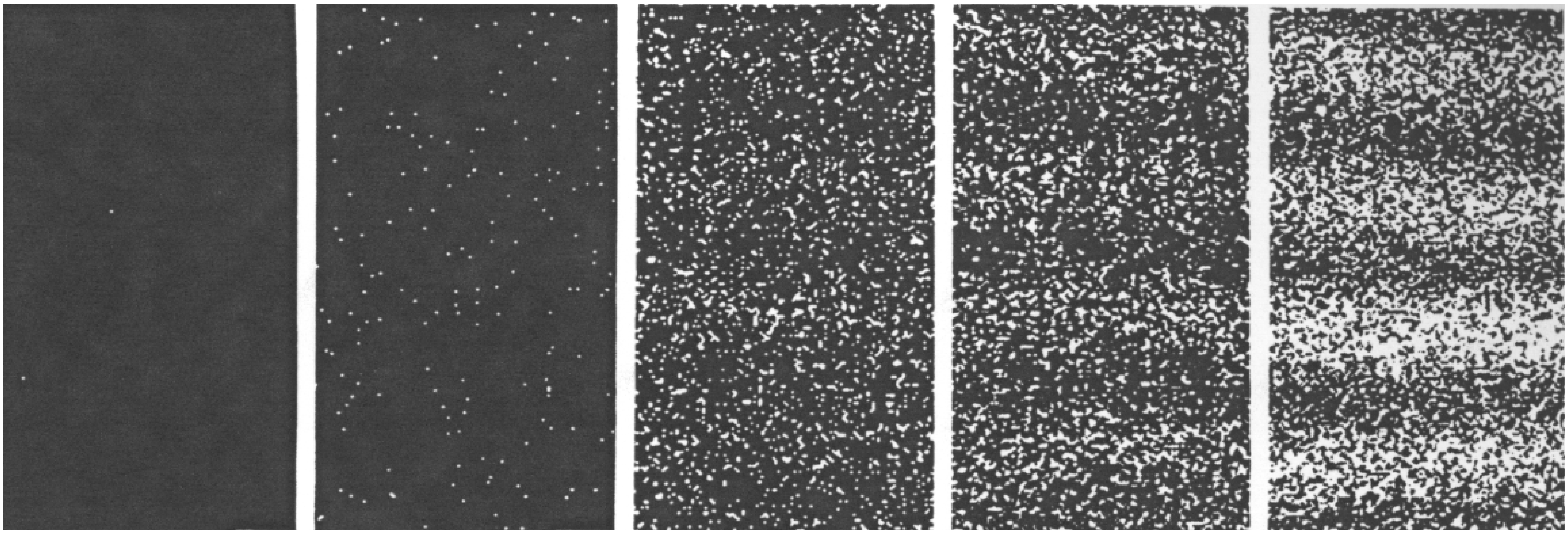
Pilot-wave theory, Bohmian metaphysics, and the foundations of quantum mechanics

Lecture 3

Elementary wave mechanics and pilot waves, with nice examples



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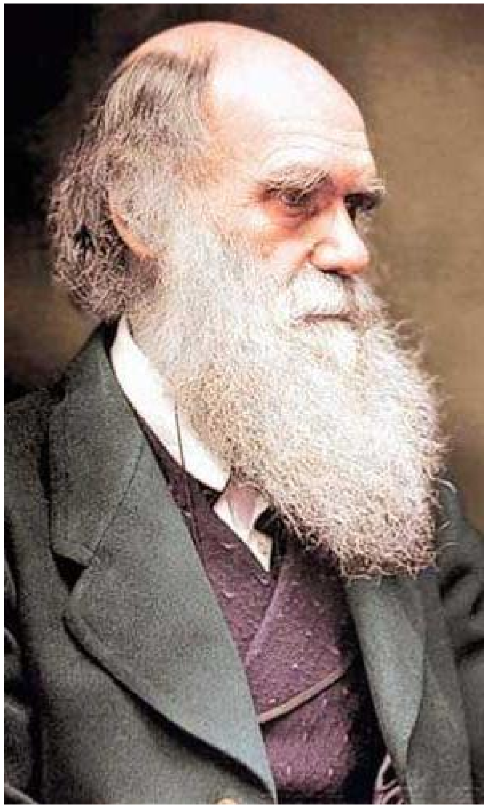
Acknowledgement

The material in this lecture is to a large extent a summary of the relevant parts of the excellent textbook *The Quantum Theory of Motion*, by Peter Holland (available on `amazon.com` or at a good book store near you). See in particular chapters 3, 4, 5 and 7. Other sources used and much other interesting material is listed on the course web page:

`www.tcm.phy.cam.ac.uk/~mdt26/pilot_waves.html`

MDT

Quantum Darwinism



Darwin theory of evolution is observer-independent, causal reconstruction of events whose random character precludes evolutionary forecasting. Historically-important issue at heart of evolution theory is *quality of its explanation* rather than its immediate predictive power. Achievement of de Broglie and Bohm in domain of quantum physical phenomena is similar in scope and conceptual structure. Yet what is perceived as intellectually satisfying and fertile mode of explanation in evolutionary biology has been fiercely resisted in quantum physics. Darwin would not fare well if judged by the criteria of prediction and control dominant in physics. But how conceive of testable prediction of explanatory theory unless we contemplate explanation it offers?

With hindsight we can now see how impractical, inhibiting ideas came to dominate and distort the entire development of quantum theory. The early quantum physicists attributed to nature a limitation we can now see was simply a deficiency of contemporary thought. [Holland, 1993]

Basic postulates of pilot-wave theory (one-body system)

1. An individual physical system comprises a wave propagating in space and time together with a point particle moving continuously under guidance of the wave.

2. The wave is mathematically described by $\psi(\mathbf{x}, t)$, a solution to Schrödinger's time-dependent wave equation (TDSE).

3. The particle trajectory is obtained as the solution $\mathbf{x}(t)$ to the equation

$$\dot{\mathbf{x}} = \left.\frac{\nabla S(\mathbf{x}, t)}{m}\right|_{\mathbf{x}=\mathbf{x}(t)}$$

   where S is the phase of ψ.

   Note this is *the probability current over the density* (in disguise). It is thus *not* an arbitrary addition to the theory - if we adopt the single semantic change that $|\psi|^2$ represents probability particle *is* at $\mathbf{x}$ rather than probability of being *found there in a suitable measurement*. An ensemble of possible motions associated with the same wave can be generated by varying initial position $\mathbf{x}_0$ - this is only info introduced not contained in $\psi(\mathbf{x}, t)$ since initial velocity fixed one you know S.

4. The probability that a particle in the ensemble lies between the points $\mathbf{x}$ and $\mathbf{x} + \mathrm{d}\mathbf{x}$ at time t is given by $R^2(\mathbf{x}, t)\ \mathrm{d}^3x$ where $R^2 = |\psi|^2$.

   Particles *need not be* so distributed, but if they are not it can be shown they will become so through relaxation to a 'quantum equilibrium' distribution and then remain so under Schrödinger evolution.

Basic equations (Bohm quasi-Newtonian reformulation)

Substitute amplitude-phase decomposition (polar form) of complex time-dependent wave function $\psi(\mathbf{x},t) = R(\mathbf{x},t)\exp(iS(\mathbf{x},t)/\hbar)$ into Schrödinger equation. Separate real and imaginary parts to get two coupled evolution equations - a continuity equation for $\rho = R^2$ and a 'quantum Hamilton-Jacobi equation' for S:

$$\frac{\partial R^2}{\partial t} + \nabla\cdot\left(\frac{R^2\nabla S}{m}\right) = 0 \quad \text{and} \quad -\frac{\partial S}{\partial t} = \frac{(\nabla S)^2}{2m} + V + Q \qquad \left[Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\right]$$

Need initial ψ for all $\mathbf{x}$ for unique TDSE solution for all t, i.e. both independent real functions $R(\mathbf{x},0)$, $S(\mathbf{x},0)$ (but better to solve *linear* wave equation directly for ψ). Can write force on particle as quasi-Newtonian $F = m\ddot{\mathbf{x}} = -\nabla(V+Q)$.

Additional new points

- ψ complex with unique value at each $\mathbf{x}$,t. Thus R single-valued fn of $\mathbf{x}$ but S not unique (up to integer multiple of $2\pi\hbar$)and undefined at nodes - can be discontinuous. Disproves all 'hydrodynamic interpretations' (e.g. Madelung, Nelson) as above equations *not equivalent* to Schrödinger without extra quantization condition: if allow multi-valued S no reason why allowed values differ by $n2\pi\hbar$ - Ψ not then single-valued. If force single-valued S exclude Ψ with multi-valued phase - e.g. those of non-zero L. [Wallstrom (1994)]. Also: Berry phases, AB effect, quantized vortices.
- Now have genuine QM *phase space* $f(\mathbf{x},\mathbf{p}) = R^2(\mathbf{x})\delta(\mathbf{p}-\nabla S(\mathbf{x}))$ with coords $(\mathbf{x},\mathbf{p})$, trajectory $\mathbf{p} = \nabla S(\mathbf{x},t)$. [Note that the well-known *Wigner function* does *not* do this].
- *Angular momentum* of particle about origin is $L(\mathbf{x},t) = \mathbf{x}\times\nabla S(\mathbf{x},t)$ evaluated along trajectory.

Basic equations (many-body system)

As fundamental theory of matter QM should apply to closed many-body system (and ultimately to universe as a whole) and reduce to theory of systems of few degrees of freedom as special case under conditions where legitimate to neglect 'rest of universe'.

For reference, many-body TDSE $i\hbar\frac{\partial\psi}{\partial t} = \left[\sum_{i=1}^{n}(-\hbar^2/2m_i)\nabla_i^2 + V(\mathbf{x}_1,\ldots,\mathbf{x}_n,t)\right]\psi$ in polar form gives usual equations with appropriate summations:

$$\frac{\partial R^2}{\partial t} + \sum_{i=1}^{n}\nabla_i\cdot\left(\frac{R^2\nabla_i S}{m_i}\right) = 0 \quad \text{and} \quad -\frac{\partial S}{\partial t} = \sum_{i=1}^{n}\frac{(\nabla_i S)^2}{2m_i} + V + Q \quad \text{with} \quad \left[Q = \sum_{i=1}^{n} -\frac{\hbar^2}{2m_i}\frac{\nabla_i^2 R}{R}\right]$$

Particle trajectories solutions $\mathbf{x}_i(t)$ to following system of n simultaneous DEs:

$$\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} = \mathbf{v}_i(\mathbf{x}_1(t),\ldots,\mathbf{x}_n(t),t) = \frac{\nabla_i S}{m_i}(\mathbf{x}_1(t),\ldots,\mathbf{x}_n(t)),t)|_{\mathbf{x}_j=\mathbf{x}_j(t)}, \qquad i,j=1,\ldots,n$$

To solve for any one trajectory need to specify initial positions of *all* particles.

Individual physical system resides in a multidimensional (configuration) space. While particles all move in 3-space, the interactions encoded in the pilot-wave irreducibly defined in $3n$ space. Generally speaking, whatever one's view of QM today, the usual spacetime framework seems too restrictive, and unable to accommodate (at least in a natural way) the phenomena associated with quantum superposition and entanglement.

Pauli : multi-dimensional config space is '*only a technical means of formulating the laws of mutual action between several particles, actions which certainly do not allow themselves to be described simply, in the ordinary way, in space and time.*' i.e. it has only mathematical significance.

Some features of many-particle systems

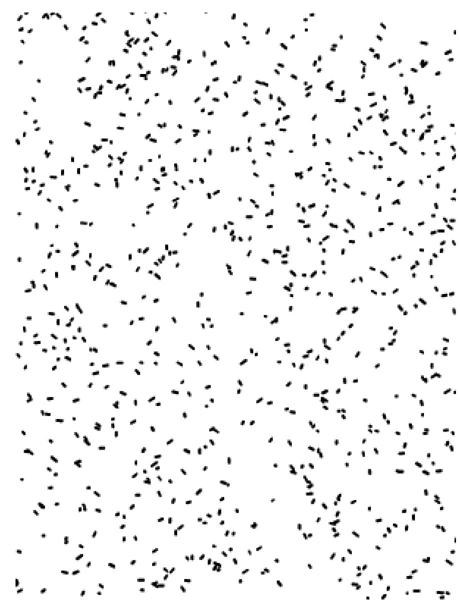
State dependence

- Force acting on particle not preassigned function of coordinates but is determined by *quantum state* of system. Infinite set of possible Q associated with same physical situation and Schrödinger equation.
- Interaction potential thus determined by something - ψ - external to (though dependent on characteristics of) particles.
- In classical dynamics, whole is sum of parts and their interactions. In QM, whole is prior to parts (particles) and its properties cannot be explained by superposition of properties of parts.

Nonlocal connection

In absence of backwards causation or many worlds, QM implies **nonlocality**, which in principle involves:

- Dependence of each particle trajectory on all others.
- Response of the whole to localized disturbances.
- Extension of actions to large interparticle distances.



Identical but distinguishable particles

•Particles **identical** if associated intrinsic params (mass, charge etc.) are same. Normally say imposing symmetry requirements on ψ of set of identical particles makes them '**indistinguishable**' in some absolute sense. Many textbooks state (with Schrödinger) that indistinguishability thus forbids particle positions. Not so!

•In pilot-wave theory, particles - guided by wave with labels removed by (anti)symmetrization - are indeed distinguishable by their individual histories.

•(Anti)symmetrization of ψ nothing to do with 'indistinguishability' but implies introduction of *forces* between particles bringing about correlations in their motion.

•Structure of Q for bosons (symmetric ψ, ignore spin) distinct from that for fermions (antisymmetric ψ). Particle correlations thus different†. Group theoretical arguments concerning topology of configuration spaces in pilot-wave theory show ψ for identical particles must be symmetric or antisymmetric (see `quant-ph/0601076` and `0506173`).

•For fermions, antisymmetry of $\psi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ implies $\psi = 0$ if two sets of coords are equal. Since config space path cannot pass through nodes, follows that two fermions cannot occupy same point in 3-space at same time (Pauli's exclusion principle). Total potential always acts in accordance with this requirement. Pauli repulsion ultimately implies exchange interaction, superexchange, Hund's rules etc..

† Differences between Maxwell-Boltzmann and Bose-Einstein statistics '*express indirectly a certain hypothesis on a mutual influence of the molecules which for the time being is of a quite mysterious nature*' [Einstein (1925) quoted in Pais biography (1982)].

Commutation relations

- Basic feature of QM: association of Hermitian operators with physical 'observables', and noncommutation relations between these operators e.g.

$$[\hat{x}, \hat{p}] \equiv \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$$

  implying that a wave function cannot simultaneously be eigenfunction of $\hat{x}$ and $\hat{p}$.

- Outcome of 'measurement' of 'observable' involves transformation of wave function into eigenfunction of associated operator†. Apparently it follows that system cannot simultaneously be in state in which position and momentum are precisely known. How reconcile with pilot-wave description where it apparently can be?

- Don't confuse *knowledge* of state of system with what state *actually is*. QM constructed so cannot 'observe' position and momentum simultaneously but this fact *per se* has no bearing on whether particle has well-defined track *in reality*.

- Think of noncommutativity here as expression of the different types of motion accessible to the particle when the wave undergoes the peculiar types of interaction appropriate to a position or momentum 'measurement'.

† Inverted commas here as we should not expect anything is actually being measured during what is usually called a 'measurement'. See next week.

Comparison with other field theories

•No 'source' of ψ-field in conventional sense of localized entity whose motion 'generates' it. ψ thus not 'radiated'.

•At this level no 'ether' introduced which would support propagation of ψ. As with electromagnetism, think of ψ as state of vibration of empty space.

•Influence of wave on particle, via Q, independent of its *intensity*.

•Initial velocity of particle fixed by initial wave function and not arbitrarily specified as in electromagnetic/gravitational theories.

•Schrödinger eqn. determines wave evolution *and* particle equation of motion (unlike electromagnetism where Maxwell equations and Lorentz force law logically distinct).

•Wave equation describes propagation of complex amplitude ψ, or equivalently two coupled real fields. Complex waves often used in other field theories for mathematical convenience, but always take real part in the end. In QM two real fields *required*.

•ψ-field finite and carries energy, momentum and angular momentum throughout space, far from where particle located (as in classical field theories). However conservation laws obeyed by field independent of particle since latter does not physically influence former.

Conditions for interference

Superposition of $\psi_1(\mathbf{x})$ and $\psi_2(\mathbf{x})$ also solution of Schrödinger eqn., with amplitude:

$$R^2(\mathbf{x}) = R_1^2 + R_2^2 + 2R_1R_2\cos\left[(S_1 - S_2)/\hbar\right]$$

Green term characterizes *interference* - finite only when component waves overlap appreciably in space. Otherwise in regions where $R_1R_2 \approx 0$ then $R^2 \approx R_1^2 + R_2^2$.

Similarly, get interference terms in the momentum field:

$$\nabla S = \frac{1}{R^2}\left\{R_1^2\nabla S_1 + R_2^2\nabla S_2 + R_1R_2\cos\left[(S_1 - S_2)\hbar\right]\nabla(S_1 + S_2) - \right.$$
$$\left. -\hbar[R_1\nabla R_2 - R_2\nabla R_1]\sin\left[(S_1 - S_2)\hbar\right]\right\}$$

If R_1R_2 do not overlap this reduces to $\nabla S \approx (R_1^2 + R_2^2)^{-1}(R_1^2\nabla S_1 + R_2^2\nabla S_2)$. Then $\nabla S \approx \nabla S_1$ or ∇S_2 depending on region of space under consideration.

•Main point is that motion of particle in overlap region is qualitatively distinct from that generated by either of component waves. Not simply a kind of 'linear superposition' of the motions generated by the partial waves.

•Note also difference between forming a *product* of wave functions (which implies physical independence of associated motions) and taking their *sum* (which implies interference and new effects if summands overlap).

Relation between particle properties and QM operators

Particle properties (e.g. E and $\mathbf{p}$) given by functions of R and S similar to expressions in classical Hamilton-Jacobi theory. What is the connection with QM operators?

Consider general Hermitian operator $\hat{A}(\hat{\mathbf{x}}, \hat{\mathbf{p}})$ which is function of position and momentum operators, and its expectation value in the state $\psi(\mathbf{x}, t) = \langle \mathbf{x} | \psi(t) \rangle$:

$$\langle \psi | \hat{A} | \psi \rangle = \frac{\int \psi^*(\mathbf{x}) \left(\hat{A}(\hat{\mathbf{x}}, -i\hbar\nabla)\psi \right)(\mathbf{x}) \ \mathrm{d}^3x}{\int \psi^*(\mathbf{x})\psi(\mathbf{x}) \ \mathrm{d}^3x} \quad \text{where} \quad (\hat{A}\psi)(\mathbf{x}) = \int \hat{A}(\mathbf{x}, \mathbf{x}')\psi(\mathbf{x}') \ \mathrm{d}^3x'$$

Hermiticity of $\hat{A}$ implies only real part of integrand contributes. Can then define **local expectation value** (LEV) of operator $\hat{A}$ in state $|\psi\rangle$ in the position representation as follows: $A(\mathbf{x}, t) = \mathrm{Re}\, \psi^*(\mathbf{x}, t)(\hat{A}\psi)(\mathbf{x}, t)/\psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)$. This is a field function of $\mathbf{x}$ and t combining information about operator and wave function - may be interpreted as a property of (an ensemble of) particles.

•For *position operator* $\hat{\mathbf{x}}(\mathbf{x}, \mathbf{x}') = \mathbf{x}\delta(\mathbf{x} - \mathbf{x}')$, LEV $\mathbf{x} = \psi^*\mathbf{x}\psi/\psi^*\psi$ for any state evaluated along a trajectory $\mathbf{x} = \mathbf{x}(t)$ *is* just the trajectory.

•For *momentum operator* $\hat{\mathbf{p}} = -i\hbar\nabla_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{x}')$, LEV $\mathbf{p}(\mathbf{x}, t) = \mathrm{Re}\, \psi^*(-i\hbar\nabla)\psi/|\psi|^2 = (\hbar/2mi|\psi|^2)[\psi^*\nabla\psi - (\nabla\psi^*)\psi] = \nabla S$ along trajectory is particle momentum $m\dot{\mathbf{x}}(t)$.

•For *Hamiltonian operator* $\hat{H} = \hat{\mathbf{p}}^2/2m + V$, LEV $E(\mathbf{x}, t) = \mathrm{Re}\, \psi^*[-(\hbar^2/2m)\nabla^2 + V]\psi/|\psi|^2 = (\nabla S)^2/2m - \hbar^2\nabla^2 R/2mR + V$ along trajectory is the total particle energy $\frac{1}{2}m\dot{x}^2 + Q + V$.

Let's introduce probability

Individual physical system comprises wave and particle which evolve entirely deterministically, but QM is a probabilistic *theory. What happened to probability?*

•In practice can never know initial conditions *precisely* even though may conceive of them as well-defined in actual fact. For real-world problems must introduce fictitious *ensemble* of systems (each element comprising a particle *and* a wave).

•Characterize ensemble with function P giving number of systems in each available state (reflects extent of our knowledge). If initial wave function one of discrete set $\psi_i(\mathbf{x}),\ i = 1, 2, \ldots$ then number of systems with initial wave function $\psi_i(\mathbf{x})$ and initial particle position in volume d^3x around point $\mathbf{x}$ given by $P_0(\mathbf{x}, i) \geq 0$

•Normalize P to unity (for all t), and $P_0(\mathbf{x}, i, t)\, \mathrm{d}^3x$ gives probability for state of an *individual* system at time t (interpreted as relative frequency of occurrence in large number of trials). Probability relates to state system actually *in* and not just to what found if measurement performed. *Ensemble average* of some physical property at definite time may be identified with average over succession of trials.

•As we shall see, to make this statistical theory correspond to QM we end up with the $|\psi|^2$-distribution of particles and the *density matrix* distribution of waves.

Introduction of probability no more intrinsic to basic theory of motion than in classical mechanics, but is postulated for practical reasons.

A statistical mechanics of waves and rays I: ensemble of particles

Since no reciprocal action of particle on wave, can assume wave and particle variables uncorrelated (independently distributed) so joint probability distribution factorizes: $P(\mathbf{x}, i) = p_i P(\mathbf{x})$*. First consider maximal knowledge of wave (fixed* ψ_i*) and partial knowledge of particles i.e. all that varies in sequence of trials is initial positions.*

•All particles associated with same wave for all t so $P(\mathbf{x}, t)$ should satisfy continuity equation. Implies normalization condition preserved for all t if true at one instant i.e. total number of particles constant in time (trajectories never come to an end).

•Since $\mathbf{p}$ is functionally related to $\mathbf{x}$ (via ∇S), probability distribution in momentum is consequence of that in $\mathbf{x}$.

•Function $P_0(\mathbf{x})$ is more or less freely specifiable. The particular choice that characterizes QM is $P_0(\mathbf{x}) = R_0^2(\mathbf{x}) = |\psi(\mathbf{x})|^2$. Good therefore that, like a probability, R^2 is ≥ 0, satisfies a conservation equation and may be normalized, and that particles cannot pass through nodes (where $R = 0$).

•Note while $P_0 = |\psi|^2$ cannot be *rigorously* justified (though see Lecture 5) it is only distribution which defines probability density on set of solution curves without preferring some point in time i.e. for other distributions the dynamics will transport the density function (according to the continuity equation) to some other function at other times. $|\psi|^2$ is in fact the natural 'equilibrium distribution' of the universe.

•Note ψ thus describes *both* the *actual* situation and our *knowledge* of that situation.

Ensemble averages and quantum-mechanical expectation values

Consider particle ensemble with wave ψ and some function $A(\mathbf{x},t)$ representing physically meaningful property of particle when evaluated along trajectory. With ensemble density $R^2(\mathbf{x},t)$ natural to define ensemble average of A at time t to be:

$$\langle A\rangle = \int R^2(\mathbf{x},t)A(\mathbf{x},t)\ \mathrm{d}^3x$$

How related to QM definition of average (expectation value) defined using Hermitian operator $\hat{A}$ corresponding to physical property $A(\mathbf{x},t)$?

$$\langle\hat{A}\rangle = \int \mathrm{Re}\ \psi^*(\mathbf{x})(\hat{A}\psi)(\mathbf{x})\ \mathrm{d}^3x$$

They are *the same* if admit actual value $A(\mathbf{x},t)$ depends on ψ as well as $\hat{A}$ and correctly identify it as *local expectation value* of $\hat{A}$. Examples:

$$\begin{aligned}
\langle\mathbf{x}\rangle &= \int R^2\mathbf{x}\ \mathrm{d}^3x = \int \psi^*\mathbf{x}\psi\ \mathrm{d}^3x = \langle\hat{\mathbf{x}}\rangle \\
\langle\mathbf{p}\rangle &= \int R^2\nabla S\ \mathrm{d}^3x = \int \psi^*(-i\hbar\nabla)\psi\ \mathrm{d}^3x = \langle\hat{\mathbf{p}}\rangle \\
\langle E\rangle &= \int R^2[(\nabla S)^2/2m + Q + V]\ \mathrm{d}^3x = \int \psi^*[-(\hbar^2/2m)\nabla^2 + V]\psi\ \mathrm{d}^3x = \langle\hat{H}\rangle \\
\langle\mathbf{L}\rangle &= \int R^2\mathbf{x}\times\nabla S\ \mathrm{d}^3x = \int \psi^*(\mathbf{x}\times -i\hbar\nabla)\psi\ \mathrm{d}^3x = \langle\hat{\mathbf{L}}\rangle
\end{aligned}$$

Expectation values as ensemble averages over a *phase space*: $\langle\hat{A}\rangle = \int R^2(\mathbf{x},t)\delta[\mathbf{p}-\nabla S(\mathbf{x},t)]A(\mathbf{x},t)\ \mathrm{d}^3x\ \mathrm{d}^3p$.

A statistical mechanics of waves and rays II: ensemble of waves

Relax requirement that quantum state precisely known - then must describe situation with a **density matrix**. For *definite* pure state ψ, expectation value is

$$\langle\psi|\hat{A}|\psi\rangle = \int \psi^*(\mathbf{x})\hat{A}(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}')\ \mathrm{d}^3x\ \mathrm{d}^3x' = \int \hat{A}(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}',\mathbf{x})\ \mathrm{d}^3x\ \mathrm{d}^3x' = \mathrm{Tr}(\hat{\rho}\hat{A})$$

where $\rho(\mathbf{x},\mathbf{x}') = \psi(\mathbf{x})\psi^*(\mathbf{x}')$ is pure state density matrix (Lecture 4). Real diagonal elements give particle distribution $\rho(\mathbf{x},\mathbf{x}) = |\psi(\mathbf{x})|^2$. Complex off-diagonal elements encode 'entanglement' or interference - if $\rho(\mathbf{x},\mathbf{x}') = 0$ then $\mathbf{x}$ and $\mathbf{x}'$ lie in non-overlapping supports (branches) of the wave function.

•If system can be in various states $|\psi_i\rangle$ with prob $p_i,\ i = 1, 2, \ldots$ (assume discrete) can use same $\mathrm{Tr}(\hat{\rho}\hat{A})$ formula if density matrix defined as $\rho(\mathbf{x},\mathbf{x}') = \sum_i p_i\psi_i(\mathbf{x})\psi_i^*(\mathbf{x}')$.

•Usually claimed average over particle ensemble fundamentally different to that over wave ensemble, but now we can use a unified approach:

$$\langle\hat{A}\rangle = \sum_i p_i \int \psi_i^*(\mathbf{x})\hat{A}(\mathbf{x},\mathbf{x}')\psi_i(\mathbf{x}')\ \mathrm{d}^3x\ \mathrm{d}^3x' = \sum_i \int P(\mathbf{x},i)A(\mathbf{x},i)\ \mathrm{d}^3x.$$

Distribution in ψ reflected in particle distribution $\rho(\mathbf{x},\mathbf{x}) = \sum_i p_i|\psi_i|^2 = \sum_i P(\mathbf{x},i)$.

•Have joint probability density for $\mathbf{x}$ and $\mathbf{p}$ ensembles (*phase space distribution*) $f(\mathbf{x},\mathbf{p}) = \sum_i p_i R_i^2(\mathbf{x})\delta(\mathbf{p} - \nabla S_i)$ obeying formula that reduces to classical Liouville equation when $Q = 0$. Since $\rho(\mathbf{x},\mathbf{x}')$ and $f(\mathbf{x},\mathbf{p})$ contain same info density matrix formalism is particular type of *statistical mechanics of waves and rays*.

Final remarks on ensembles

1. As density matrix ρ represents *fictitious ensemble* all waves considered to occupy - simultaneously - overlapping regions of space *without interfering*. Only one wave and one particle present in any one trial. Density matrix thus describes both an 'ensemble of ensembles of particles', and an ensemble of waves and particles.
   - Note that when the component waves do not physically coexist, one says we have a *proper mixture*.
   - In an *improper mixture*, the components physically coexist but do not overlap.
2. Essential difference between this notion of probability and the one usually employed:
   - Pilot-wave theory describes likely state of matter as it actually is, whatever processes it may be part of. Both R^2 and ρ refer to our partial knowledge of system which is in itself well-defined.
   - In Born interpretation $|\psi|^2$ does not represent our ignorance of an actual state but concerns distribution of values found if one performs a '*measurement*'. In pilot-wave theory we recover Born's interpretation as a special case for the particular processes characterized as measurements (Lecture 4).

Eigenvalues, probabilities, and measurements

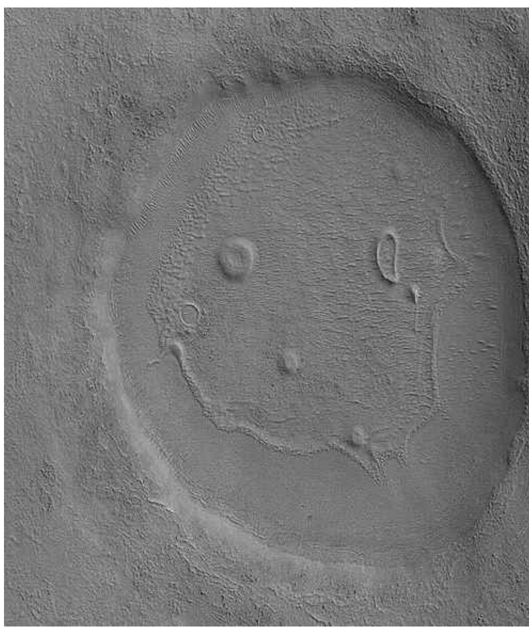
•In standard QM determine state of system (e.g. momentum) through process part of whose outcome is that system is left in eigenstate of associated operator.

•System then said to 'have' definite momentum (eigenvalue of momentum operator) with conjugate variable $\mathbf{x}$ being completely unknown, randomly fluctuating, undefined, or whatever (!). (What actually *is* it that has this well-defined attribute?).

•How do *actual values* in pilot-wave theory (well-defined for all states, continuously variable) relate to *eigenvalues* of corresponding operators ('only definite properties a system may possess')? If system left in eigenstate, actual value *coincides* with eigenvalue e.g. $L_z = x\partial_y S - y\partial_x S \rightarrow m_z\hbar$ when $\psi(\mathbf{x}) \rightarrow \mathrm{e}^{im_z\phi}$ with $\hat{L}_z\psi = m_z\psi$. 'Measurements' just rather ordinary interactions which typically occur all the time during which actual values evolve continuously into appropriate eigenvalue.

•Suppose $\psi(\mathbf{x}) = \sum_a c_a \psi_a(\mathbf{x})$ with $\psi_a(\mathbf{x})$ eigenfunctions of operator, eigenvalues a. *Interference* between summands implies sum *not* ignorance of current state as with proper mixture. Aim of measurement to separate the ψ_a (by coupling to another system) so they no longer overlap. Then superposition behaves as *if* it were a mixture. Then actual value $A(\mathbf{x}) = \mathrm{Re}\ \psi^* \hat{A}\psi/|\psi|^2 = a$ in domain where ψ_a is finite. Hence $g(a) = \int |\psi(\mathbf{x})|^2 \delta[a - A(\mathbf{x})]\ \mathrm{d}^3x = |c_a|^2$ is prob that system in state ψ will be *found* in state ψ_a (**Born's postulate** - special case of prob that it *is* in certain state).

Let's do some examples!



Classically free particles

When $V = 0$ the particle equation of motion is $m\ddot{\mathbf{x}} = -\nabla Q|_{\mathbf{x}=\mathbf{x}(t)}$. Classically free motion thus not free in QM since for generic solutions of Schrödinger equation Q is non-vanishing. Motion depends on choice of wave function. Contrasts with uniform rectilinear motion implied by classical mechanics.

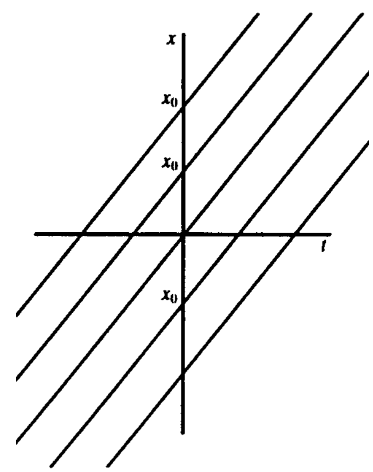
Plane-wave ψ (momentum eigenstates)

$\psi(\mathbf{x},t) = A(t)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}$ eigenfunction of momentum operator with eigenvalue $\hbar\mathbf{k}$. $\psi(\mathbf{x},t) = \frac{1}{L^{3/2}}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}$ stationary state soln of wave eqn for $V = 0$ and $\omega_{\mathbf{k}} = \frac{\hbar\mathbf{k}^2}{2m}$.

•Constant amplitude $\implies Q = 0$. 'Classical'-type wave obeying classical HJ eqn.

•Phase $S(\mathbf{x},t) = \hbar\mathbf{k}\cdot\mathbf{x} - \hbar\omega_{\mathbf{k}}t$. Constant S wavefronts are planes propagating in $\mathbf{k}$-direction with speed $\omega_{\mathbf{k}}/|\mathbf{k}| = \hbar|\mathbf{k}|/2m$. Particle trajectories straight lines orthogonal to S surfaces. Guidance eqn $\mathbf{p} = \nabla S = \hbar\mathbf{k}$ and $E = -\partial S/\partial t = \hbar\omega_{\mathbf{k}}$ (de Broglie!).

$$\boxed{\text{Trajectory: } \mathbf{x}(t) = \mathbf{x}_0 + \hbar\mathbf{k}t/m}$$



•In momentum eigenstate normally say definite momentum but position completely unknown (or even 'does not exist'). In pilot-wave theory particle has well-defined position and uncertainty relation doesn't have implication normally ascribed to it.

Heisenberg's relations and uncertainty

$$\boxed{\text{HUR: } \Delta x_i \Delta p_j \geq \frac{\hbar}{2}\delta_{ij}}$$

What does this mean? Three distinct interpretations can be supported, relating to:

1. **Wave structure**: (classical) wave train width in physical space and wave number spread in Fourier space connected by $\Delta x_i \Delta k_i \approx 1$ then $\mathbf{p} = \hbar\mathbf{k}$ yields HUR. Obvious.
2. **Properties of particle ensemble**: Limitation on scatter in results of statistical ensemble of identical experiments. Standard state prep, measure $\mathbf{x}$ (precisely) many times $\longrightarrow \Delta\mathbf{x}$, then $\mathbf{p}$ (precisely) many times $\longrightarrow \Delta\mathbf{p}$, $\Longrightarrow$ HUR. This is *operational definition* of HUR i.e. how one would test it experimentally.
3. **Current behaviour of single particle**: Ensemble definition not about *simultaneous measurement* i.e. can we attribute simultaneously well-defined $\mathbf{x}$ and $\mathbf{p}$ coords. Yet Heisenberg *et al.* claim application of HUR to *individual cases* relevant to this, and that no meaning to notion of spacetime trajectory (*oh dear!*) as more precise position entails corresponding loss in determination of momentum in 'unsharp measurements'.

Definition 3 *not clear*. Is it that we cannot possess precise simultaneous knowledge of things that are in themselves well-defined (*out-of-focus photo*)? Is there a fuzziness in their definition (*sharp photo of fog*)? Or do they have no meaning? Whatever. Heisenberg's argument assumes $\Delta\mathbf{p}$ refers to knowledge uncertainty of **current particle momentum** in imprecise $\mathbf{x}$ or $\mathbf{p}$ determination. *Not justified!* Traditional argument against trajectory concept (led to view that classical conception of material systems must be replaced by more nebulous view of physical reality) is demonstrably incorrect.

Heisenberg's relations and 'uncertainty': pilot-wave perspective

How do we define the uncertainty in **current particle momentum** *for pilot waves?.*

In pilot-wave theory actual momentum $\mathbf{p} = \nabla S(\mathbf{x})$ unknown only because the position is. Distribution of true momentum in state $\psi = R\mathrm{e}^{iS/\hbar}$ is

$$g(\mathbf{p}) = \int R^2(\mathbf{x})\delta[\mathbf{p} - \nabla S(\mathbf{x})]\,\mathrm{d}^3x$$

so mean square deviation from mean of x-component of actual momentum given by

$$(\Delta p_x^{PWT})^2 = \int g(\mathbf{p})p_x^2\,\mathrm{d}^3p - \langle p_x\rangle^2 = \int R^2(\mathbf{x})(\partial S/\partial x)^2\mathrm{d}^3x - \left(\int R^2(\mathbf{x})(\partial S/\partial x)\,\mathrm{d}^3x\right)^2.$$

Same expression as for single-valued classical ensemble (though ψ satisfies the Schrödinger equation) thus at any t this gives measure of uncertainty in our knowledge of p_x corresponding to uncertainty Δx in our knowledge of x. Conclude that

$$\boxed{\Delta x \Delta p_x^{PWT} \geq 0}$$

i.e. there is no particular reciprocal limit on the precision with which position and momentum may be known or specified, according to this definition of uncertainty.

So what is Heisenberg's 'momentum uncertainty' $\Delta\hat{p}_x$?

What meaning can be attributed to $\Delta\hat{p}_x$ in pilot-wave theory, apart from measure of dispersion in results of precision momentum measurements?

•At instant $\psi(\mathbf{x})$ is formed have $(\Delta\hat{p}_x)^2 = \int (p_x - \langle\hat{p}_x\rangle)^2 |\phi(\mathbf{p})|^2 \ \mathrm{d}^3p$. Now $|\phi(\mathbf{p})|^2$ doesn't refer to distribution in actual $\mathbf{p}$ in ensemble so this doesn't represent uncertainty in actual $\mathbf{p}$. Representing $\psi(\mathbf{x})$ as linear sum of momentum eigenfunctions doesn't imply ensemble is mixture of particles having momentum $\mathbf{p}$ with relative frequency $|\phi|^2$ because of interference between Fourier components. Actual $\mathbf{p}$ defined by ∇S - may be totally different from that of any component plane wave.

•Heisenberg momentum uncertainty can be identified with component of total stress tensor of the ψ-field through $(\Delta\hat{p}_x)^2 = (\Delta p_x^{PWT})^2 - \hbar^2 \int R(\partial^2 R/\partial x^2) \ \mathrm{d}^3x$ (see Holland, Ch.8). Gives info on current mean value of this as a particle property - not the actual momentum. Origin of statistical correlations between $\mathbf{p}$ and $\mathbf{x}$ measurements due to distribution of stresses in ψ-field (which arise since field guiding particle in ensemble also enters into definition of mean values).

•$|\phi(\mathbf{p})|^2$ gives prob of outcome p_x and $\Delta\hat{p}_x$ preserved by measurement interaction (checking transforms $\psi(\mathbf{x}) \rightarrow \mathrm{e}^{ip_x/\hbar}$). $\Delta\hat{p}_x$ is measure of accuracy with which can predict the outcome of a subsequent $\mathbf{p}$ measurement for particle in state $\psi(\mathbf{x})$.

•Heisenberg's inequalities *not relevant* to issue of whether matter may be attributed objective properties such as simultaneously well-defined $\mathbf{x}$ and $\mathbf{p}$ variables.

Stationary states

$$\boxed{\Psi(\mathbf{x},t) = \Psi_0(\mathbf{x})e^{-iEt/\hbar}}$$

Eigenfunctions of the Hamiltonian $\hat{H} = \frac{-\hbar^2}{2m}\nabla^2 + V$ with V independent of time. In $\Psi = R\exp(iS/\hbar)$ form we have $R(\mathbf{x},t) = R_0(\mathbf{x})$ and $S(\mathbf{x},t) = S_0(\mathbf{x}) - Et$.

Consequences

•Probability density is independent of the time: $|\Psi|^2 = R_0^2(\mathbf{x})$ i.e. no time-dependence to where particle is likely to be found - there is effectively *no motion*.

•Quantum potential $Q = (-\hbar^2/2mR_0)\nabla^2 R_0$ is time-independent. Therefore so is the total effective potential $\partial(V+Q)/\partial t = 0$.

•The velocity field is independent of time. If $\Psi_0(\mathbf{x})$ is a real function then the velocity is *zero* (Recall probability current $\mathbf{j} = \frac{\hbar}{2mi}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)$). The particle is at rest where one would classically expect it to move since the quantum force $(-\nabla Q)$ cancels the classical force $(-\nabla V)$.

•The energy of all particles in the ensemble, $-\partial S/\partial t$, is a constant of the motion and equal to the energy eigenvalue E.

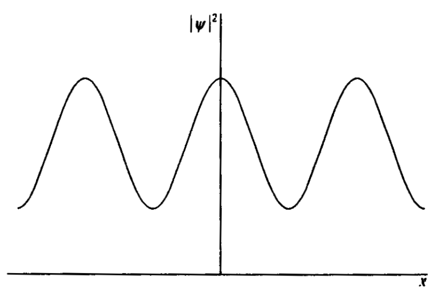
Wave packets

•Any linear combination of stationary solutions to TDSE also solution. Each term in sum has its own t-dependent phase factor giving overall t-dependence in prob density. For purposes of this course a *wave packet* is just a *superposition of states having different energies* (and thus a t-dependent $|\psi|^2$). For discrete or continuous spectra:

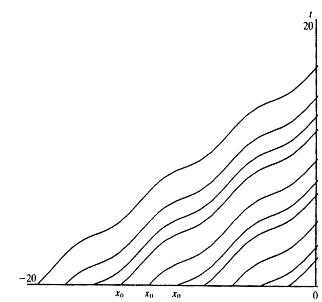
$$\psi(x,t) = \sum_{n=1}^{\infty} a_n \psi_n(x) e^{-\frac{i}{\hbar}E_n t} \quad \text{or} \quad \psi(x,t) = \int_0^{\infty} a(E)\psi_E(x) e^{-\frac{i}{\hbar}Et}\,\mathrm{d}E$$

•Note t-dependence comes exclusively from interference term, e.g.

$$|\psi(x,t)|^2 = |a_1|^2|\psi_1(x)|^2 + |a_2|^2|\psi_2(x)|^2 + 2\mathrm{Re}\left\{a_1^* a_2 \psi_1^*(x)\psi_2(x) e^{-i\frac{(E_2-E_1)t}{\hbar}}\right\}$$

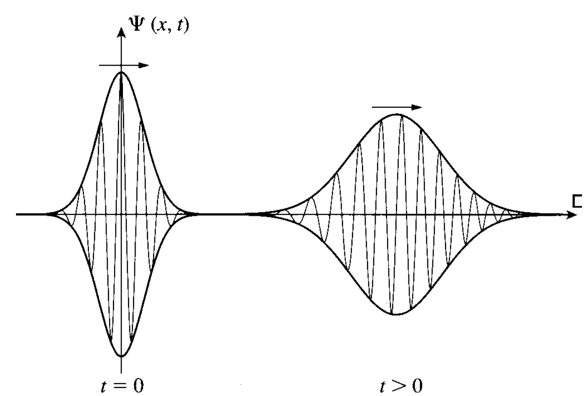


Example: superposition of 2 plane waves with $\mathbf{k} \neq \mathbf{k}'$ gives $|\psi|^2$ in 1D like this (moving to right) with strict nodes only if partial waves have same amplitude. Particle trajectory oscillates symmetrically about line $\omega t + a$.



•Usually *wave packet* implies confining particle to region. Superpose many plane waves and arrange amplitudes and phases so constructively interfere in restricted region, and destructively interfere outside it.

$$\psi(\mathbf{x},t) = (2\pi)^{-\frac{3}{2}} \int \phi(\mathbf{k}) \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)}$$



Tunnelling and the potential step

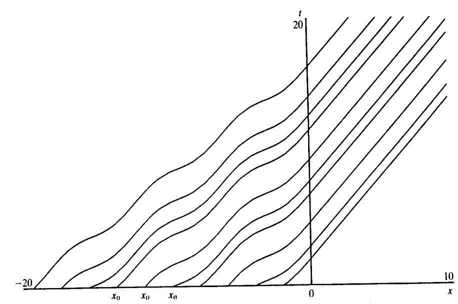
Tunnelling : the appearance of particles in ‘classically forbidden regions’. Usual discussions employ plane waves incident on idealized potentials with simple shapes. Reflection/transmission coefficients R and T apparently give insight into tunnel effect.

1D potential step of height V

$E > V$: Non-classical feature apparent possibility of particle reflection by potential.

$$\psi(x,t) = \begin{cases} A(\mathrm{e}^{ikx} + c\mathrm{e}^{-ikx})\mathrm{e}^{-iEt/\hbar}, & x < 0 \\ Ab\mathrm{e}^{i(k'x - Et/\hbar)}, & x > 0 \end{cases} \quad \Longrightarrow \quad \begin{aligned} R &= c^2 = \frac{(k-k')^2}{(k+k')^2} \\ T &= b^2\frac{k'}{k} = \frac{4kk'}{(k+k')^2} \end{aligned}$$

For $x < 0$ trajectory oscillates about straight line and total E conserved with classical value (but KE and Q variable). Beyond step, trajectory straight with classical $v = \hbar k'/m$. No reflection but $R \neq 0$ and $T \neq 1$!



$E < V$: Finite exponentially decaying prob of particle in classically forbidden domain $x > 0$. Particle here *at rest* with negative $Q = E - V$. For $x < 0$ also at rest with $E = Q (= \hbar^2 k^2/2m)$. Coefficients $R = 1$, $T = 0$ - not what particle actually does!

Reflection/transmission coefficients do not therefore define attributes of the actual motion. Notions that the particle is ‘incoming’ or ‘reflected’ do not apply. Why?

Tunnelling and the potential step II

Plane waves not a good idea..

- Nothing in plane wave description of step corresponds to initially-free incident particle that after finite time encounters step and is either reflected or transmitted.

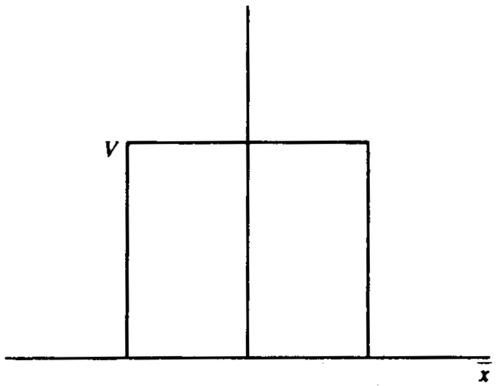
- Reflected wave does not gradually form as consequence of scattering by step but rather is *already defined* for all x at $t = 0$, on an equal footing with incident wave. Superposition has entirely different properties to either summand.

- At $t = 0$ wave already accommodated to boundary conditions at $x = 0$ and so particle wherever initially placed moves under influence of step. It is never 'free'. Also $|\psi|^2$ finite for all x and $t = 0$ so x_0 could lie in $x > 0$ ('transmission region').

- The plane wave theory thus fails to conform to usual description of tunnelling processes and mental images we may harbour about them. If want to start with initially free system incident on barrier need to employ *wave packets*.

- In $E < V$ case state of rest not inconsistent with finite momentum which would be found in an experiment. Need to continually emphasize that results of a measurement - though causally and continuously connected with premeasurement value - in general differ from the latter due to disturbance caused by interaction with measuring device.

Tunnelling through a square barrier

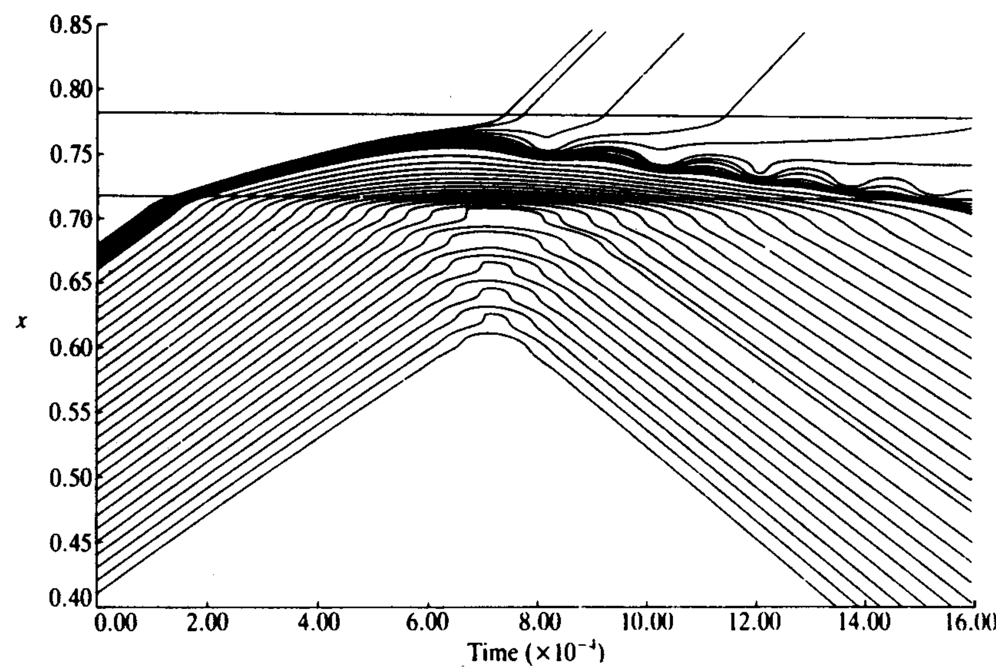


More realistic description: 1D scattering of Gaussian packet of mean energy E from square barrier $V > 0$. Interaction of packet with barrier leads to formation of reflected and transmitted packets of diminished amplitude,perhaps together with small packet that may persist inside barrier for some time. Particle ends up in one of these.

Consider $E < V$. Tunnelling arises from modification of total energy of particle (initially $\approx E$) due to rapid spacetime fluctuation of ψ-wave in vicinity of barrier. Total particle energy $-\partial S/\partial t = (1/2m)(\partial S/\partial x)^2 + V + Q$ evaluated along trajectory.

Effective 'barrier' encountered by particle is not V but $V + Q$ - may be higher or lower than V and may vary outside 'true' barrier. For tunnelling require only that $-\partial S/\partial t \geq V + Q$ then particle may enter or cross barrier region.

Impossible to explain this effect consistently using interpretation involving wave function alone.



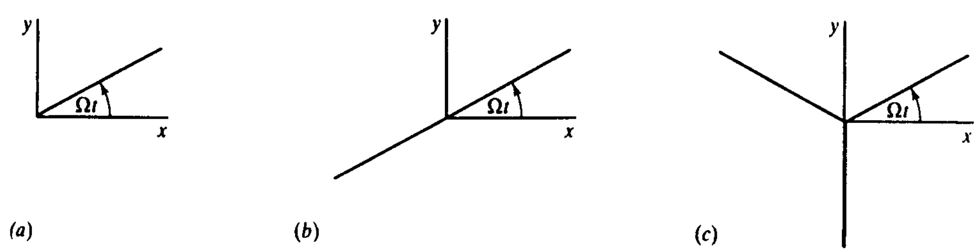
Hydrogen-like atom

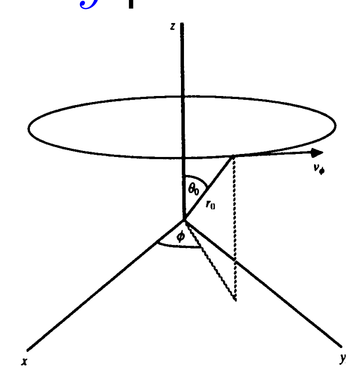
Rotating plane wave and electron trajectories

For H atom, first examine rotational analogue of linear translation of momentum eigenfunction wavefronts i.e. look at eigenfunctions of orbital angular momentum component (say $\hat{L}_z$). Need simultaneous eigenfunctions of $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ as $[\hat{L}_z, \hat{\mathbf{L}}^2] = 0$.

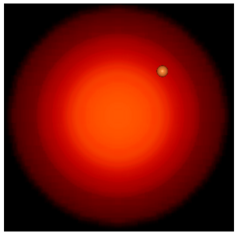
•In spherical polar coordinates $\hat{L}_z = -i\hbar\partial/\partial\phi$ and $\hat{\mathbf{L}}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$. Eigenvalue eqns $\hat{L}_z Y_{lm}(\theta,\phi) = m\hbar Y_{lm}(\theta,\phi)$ and $\hat{\mathbf{L}}^2 Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 Y_{lm}(\theta,\phi)$ where Y_{lm} are spherical harmonics, $l = 0, 1, 2, \ldots$ is **orbital angular momentum quantum number**, and $m = -l \leq m \leq l$ is **azimuthal quantum number**. $Y_{lm}(\theta,\phi) = f_{lm}(\theta)\mathrm{e}^{im\phi}$ where f_{lm} is set of real functions proportional to Legendre polynomials.

•For TDSE solution assume system free or spherically symmetric $V(r)$ so wave eqn separates: $\psi_{Elm}(r,\theta,\phi) = g_{Elm}(r) f_{lm}(\theta)\mathrm{e}^{i(m\phi - Et/\hbar)}$ with real g. Phase function $S(r,\theta,\phi,t) = m\hbar\phi - Et$. For each t and $m \neq 0$ constant S wavefronts planes parallel to/ending on z-axis. Planes rotate about z-axis with angular velocity $\Omega = E/m\hbar$. No of wavecrests defined by $S = n\hbar$ that come to an end on z-axis equal to $|m|$. Trajectories orthogonal to wavefronts i.e. *circles* in planes parallel to xy-plane.

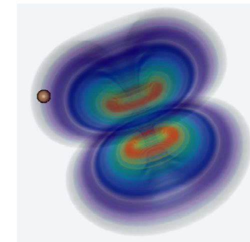




Hydrogen-like atom



•$\hat{L}_z$ *eigenfunction*: particle orbits z-axis, constant speed and radius (freely specifiable independent of E or m with $P = |\psi|^2 r^2 \sin\theta \; \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$). Exactly $|m|$ de Broglie λ in one quantum orbit. Unlike primitive Bohr model where e^- in equatorial plane and radius function of m (PWT has velocity quantization: higher $m \Rightarrow$ faster particle).



•If ψ eigenfunction of $\hat{L}_z$, $\hat{\mathbf{L}}^2$ actual values and eigenvalues coincide. Traditionally x- and y-components 'undefined' but in fact well defined: $L_x = -m\hbar\cot\theta\cos\phi$, $L_y = m\hbar\cot\theta\sin\phi$, $L_z = m\hbar$. Along trajectory L_x, L_y *not conserved* (unlike $\hat{L}_z$, $\hat{\mathbf{L}}^2$). Effective potential thus not symmetric (for symmetric V).

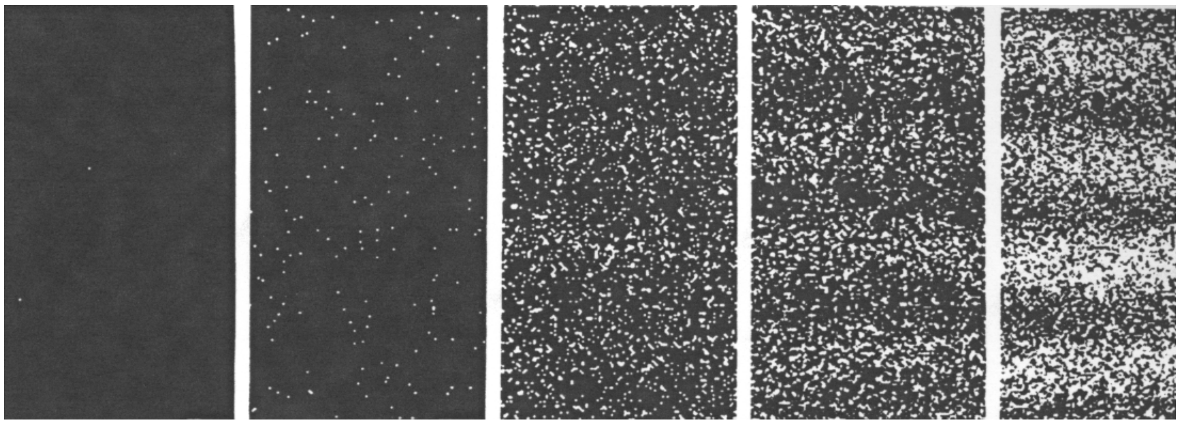
•Arbitrary V and unrestricted E so far. In H-like atom have $V = -Ze^2/r$ and $E_n = -m_0 Z^2 e^4/2\hbar^2 n^2$. Most general stationary state of given n is

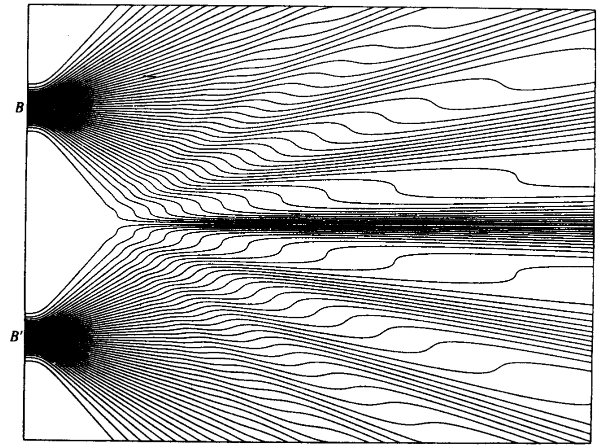
$$\psi_n(r,\theta,\phi,t) = \sum_{l=0}^{n-1}\sum_{m=-l}^{l} c_{lm}\psi_{nlm}(r,\theta,\phi,t) = \left(\sum_{l=0}^{n-1}\sum_{m=-l}^{l} c_{lm}F_{nl}(r,\theta)\mathrm{e}^{im\phi}\right)\mathrm{e}^{-iE_nt/\hbar}$$

Phase of ψ_n complicated function of r, θ, ϕ. Trajectory has correspondingly complex structure (of constant E). Arbitrary $c_{lm} \Rightarrow$ infinite set of possible motions.

•For $m = 0$ then Q balances V and e^- at rest relative to nucleus. True even if $l > 0$ i.e. finite L^2 though not moving (L^2 expression contains term not connected with motion). Quantum numbers do not directly represent dynamical properties!

Interference: two-slit experiment





Empirical manifestation of the wave aspect of matter is a statistical effect resulting from aggregation of discrete single-particle processes. 'Wave-particle' duality cannot mean matter manifests itself either as a wave or particle (in their classical senses) depending on experimental arrangement. 'Wave' only ever *made apparent by observation of particle positions.*

•**Pilot-wave interpretation**: Localized particle concept correct (see localized emission and detection process) but quantum particle follows spacetime track different from classical particle. Need extra force. Postulate objective wave - involved in *each* process - which passes through both slits, interferes with itself, and 'guides' the particle. Observed pattern builds up over many trials as $|\psi|^2$. Physical meaning obscure if ψ just 'probability wave', knowledge, information, or applies only to ensembles.

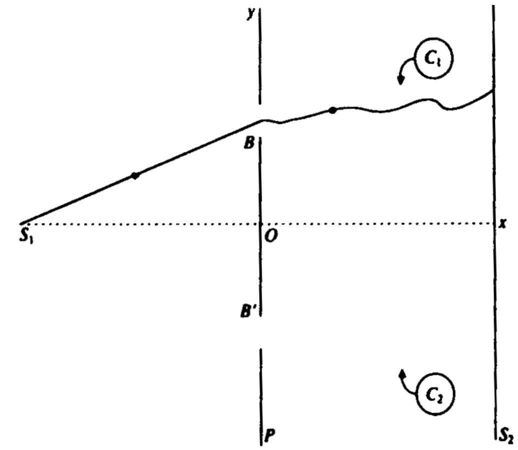
•**More detailed predictions than contained in wavefunction**: (1) Each particle passes through one slit or the other, (2) Wave function single-valuedness implies no trajectory can cross or even intersect apparatus axis of symmetry - thus know which slit from on-screen particle position.

•Can show that as expected insertion of probe to check above predictions will modify wave and disturb trajectories so as to wash out interference pattern (Holland p. 374). Usual 'which path' discussions of this in standard QM (i.e. within pure wave function formalism) meaningless in all circumstances.

‘Delayed-choice’ experiments

Example of paradoxes that arise if subdivide phenomena without adopting consistent model describing what an electron is and how it behaves in an interferometer.

Basic experiment: Usual two-slit experiment but with two counters that can be quickly inserted into two arms of interferometer between slits and detecting screen in time much shorter than electron takes to traverse this region. Counters reveal ‘which path’ electron took. Main point: decision on whether to insert counters (determine which path) or to leave them out (and let electron contribute to interference pattern i.e. take 'both paths') can be made *after* electron has ‘passed’ slit plane.



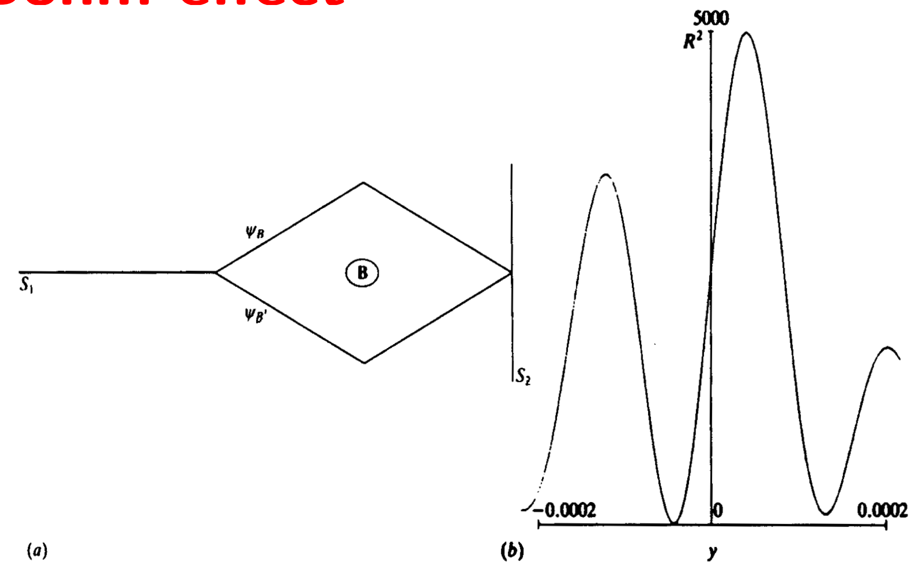
Standard argument: Suggests earlier behaviour of electron (passage through one or both slits) influenced by later decision whether to insert counters. To avoid paradoxical conclusion that electron somehow traversed both slits *and* just one slit, Wheeler proposed past has no existence except as recorded in the present. Phenomenon of electron passage ‘is not a phenomenon until it is an observed phenomenon’. In other words, confusion avoided only if we desist from analyzing functioning of device.

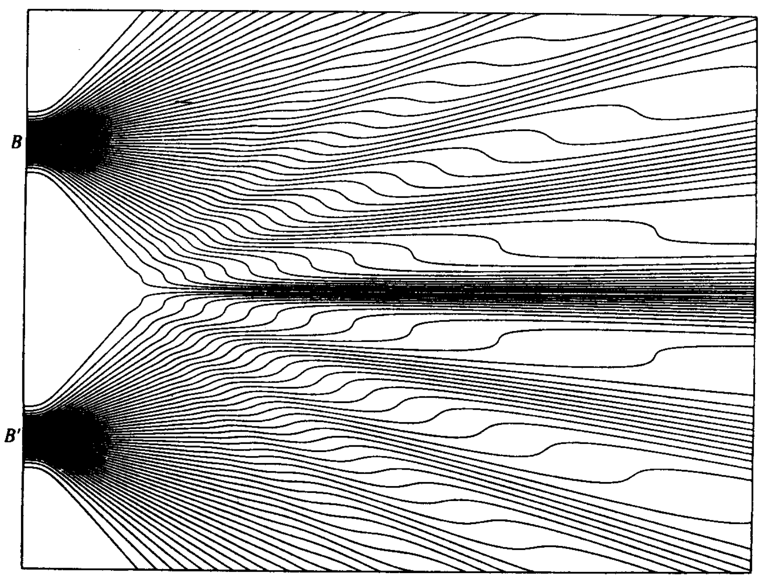
Pilot-wave resolution: Above conclusion trivially sidestepped - argument taken simply as evidence that wrong model of individual physical system has been employed (or rather that no serious attempt has been made to develop one). Passage of particle through one slit and wave through both forms well-defined time-dependent physical process. What happens later has no bearing at all. If both paths open particle responds to overlapping waves. Detecting plate reveals this but doesn’t influence it. If counters inserted prior to wave overlap, evolved total wave different and in detecting particle the counters simply reveal this. *The present merely reveals the past and has no influence on it.*

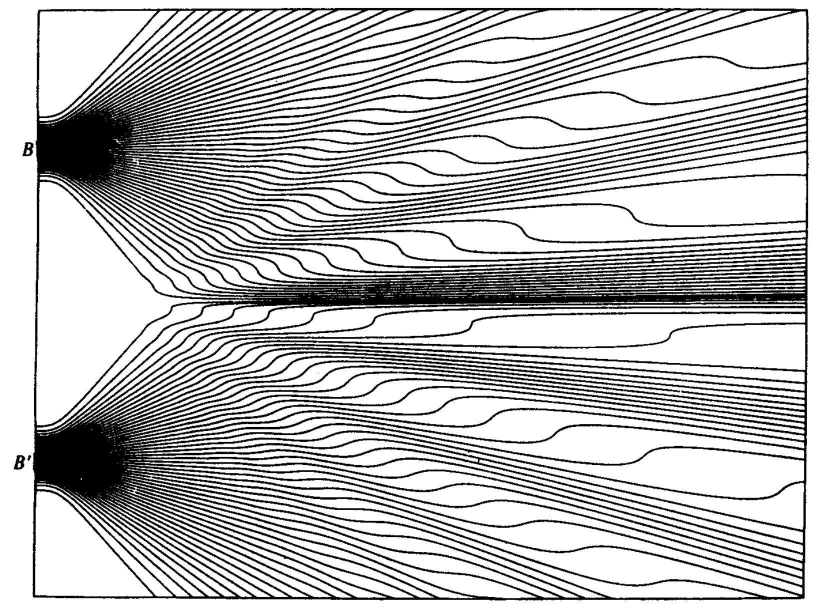
Aharanov-Bohm effect

Effect concerns existence of electromagnetic influences on interfering charged particle beams that are confined to spacetime regions containing no electric or magnetic fields.

Basic experiment: Coherent electron beams passing on either side of an inaccessible magnetic field $\mathbf{B}$ experience a relative phase shift that is aperiodic function of the flux.







Ψ picks up path-dependent phase factor $e^{ie/c\hbar \int \mathbf{A}\cdot \mathbf{dx}}$. With pilot waves Q (i.e. Ψ) carries info on potentials not present in field strengths. Change in interference pattern leads to (gauge-invariant) redistribution of particle trajectories. Lorentz force $\mathbf{F}$ does not exhaust possibilities for electromagnetic field to influence motion - the quantum force may be finite in region where $\mathbf{F} = 0$! [Holland p.124, p. 190]

Are there quantum jumps?

Standard belief: *systems can only possess certain values of physical quantities corresponding to spectra of Hermitian operators*.

•Not true in pilot-wave theory (surprise!). Quantities well-defined and continuously variable for all quantum states - values for subset of eigenstates have no *fundamental* physical significance. One of characteristic features of QM - existence of discrete energy levels - due to *restriction of basically continuous theory to motion associated with subclass of eigenfunctions*. Such states may possess particular physical importance in relation to stability of matter, but particle momentum and energy just as unambiguously defined when wave is superposition of eigenstates. There are no 'quantum jumps' in sense of process that is instantaneous or beyond analysis.

Example: collision of electron and hydrogen atom

Transition of H atom from GS to excited state due to inelastic collision with electron (Franck-Hertz experiment). Discrete change in energy understood as outcome of basically continuous process, with actual energy change uniquely determined by initial positions of atomic and incident electrons.

Initial conditions:

Atomic electron: At rest, coordinate $\mathbf{x}$ in ground state $\psi_0(\mathbf{x})\mathrm{e}^{-iE_0t/\hbar}$, total energy E_0.
Incident electron: Coordinate $\mathbf{y}$, free wave packet $F(\mathbf{y},t)=\int f(\mathbf{k}-\mathbf{K})\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{y}-\hbar\mathbf{k}^2t/2m)}\,\mathrm{d}^3k$ with $f(\mathbf{k}-\mathbf{K})$ peaked around $\mathbf{K}$. Centre of packet moves along path $\mathbf{y}=\hbar\mathbf{K}t/m$ towards atom. Variable energy of approximately $\hbar^2\mathbf{K}^2/2m$.

Whilst incident packet and atom separated in space and do not overlap the total wave function is a product: $\Psi_i(\mathbf{x},\mathbf{y},t)=\psi_0(\mathbf{x})\mathrm{e}^{-iE_0t/\hbar}F(\mathbf{y},t)$. Motions of $\mathbf{x}$- and $\mathbf{y}$-electrons initially independent.

Collision of electron and hydrogen atom

As packet approaches atom, electrons interact with one another and with nucleus. Expand Ψ in complete set of atomic eigenfunctions $\psi_n(\mathbf{x})$ (including discrete and continuous bits of spectrum):

$$\Psi(\mathbf{x},\mathbf{y},t) = \Psi_i(\mathbf{x},\mathbf{y},t) + \left(\sum_n + \int\right) \psi_n(\mathbf{x}) \mathrm{e}^{-iE_n t/\hbar} f_n(\mathbf{y},t)$$

System point - initially in region of config space where Ψ_i appreciable - now influenced by interference of Ψ_i and scattered waves $\psi_n(\mathbf{x}) f_n(\mathbf{y})$ in region where overlap. The $\mathbf{x}$ and $\mathbf{y}$ motions become closely correlated (and complicated in general). For long t asymptotic form of this is:

$$\Psi = \Psi_i + \left(\sum_n + \int\right) \psi_n(\mathbf{x}) \mathrm{e}^{-iE_n t/\hbar} \int f(\mathbf{k}_n - \mathbf{K}_n) r^{-1} \mathrm{e}^{i(\mathbf{k}_n \cdot \mathbf{y} - \hbar \mathbf{k}_n^2 t/2m)} g(\theta, \phi, \mathbf{k}_n)\; \mathrm{d}^3 k_n$$

with $y = (r, \theta, \phi)$ and $\hbar^2 \mathbf{k}_n^2/2m + E_n = \hbar^2 \mathbf{K}^2/2m + E_0$ for each n. This is sum of outgoing packets each correlated with atomic eigenfunction $\psi_n(\mathbf{x})$. These eventually will cease to overlap i.e. Ψ becomes superposition of nonoverlapping config space functions (statistically equivalent to *mixture*)

System point moves into one outgoing packet, definite outcome achieved, *effective wave function* is one term in above sum. Factorizable into fn of $\mathbf{x}$ times fn of y so electrons again independent with well-defined energies. Atom has 'jumped' to nth stationary state absorbing discrete energy $E_n - E_0$. Outgoing particle free with KE $\hbar^2 \mathbf{K}_n^2/2m$. Outcome of entirely continuous (but rapid) process that *appears* discontinuous but atomic electron moved from rest to uniform circular motion via unstable but well-defined trajectory. 'Reoverlap' with empty waves overwhelmingly unlikely, particularly if 'measure' energy to determine final state of atom - a process which involves amplification to macroscopic scale.

Aftermath

Pilot-wave theory reproduces *all* predictions of elementary wave mechanics but adds an explanation in terms of objectively existing particles and their motions (though not always in the way you expect). This is despite the fact that most people (have been led to) believe this is physically and logically impossible. Whatever else you might think about that - it's certainly very interesting!

Antony Valentini email to MDT:

"Some of the questions that were raised after [your earlier talk] sound like the following familiar variety: because you're talking about particle trajectories, people start thinking about the whole thing in terms of classical physics, and raise objections that are based on classical assumptions. ... There is a tendency to insist, wrongly, that the theory must respect certain features of classical physics (Holland points this out in his book, noting that some people in effect complain that the theory is 'not classical enough'). A lot of people think about quantum physics in terms of informal semiclassical pictures, and when they are told about pilot-wave trajectories they take it as an invitation to take those semiclassical pictures seriously. They basically need to be told that this is a new form of dynamics, with quite new laws of motion, so that objections based on classical assumptions just do not apply."

"IT'S UNIFIED AND IT'S A THEORY, BUT IT'S NOT THE UNIFIED THEORY WE'VE ALL BEEN LOOKING FOR."

Rest of course

Lecture 1: 21st January 2009
An introduction to pilot-wave theory

Lecture 2: 28th January 2009
Pilot waves and the classical limit. Derivation and justification of the theory

Lecture 3: 4th February 2009
Elementary wave mechanics and pilot waves, with nice examples

Lecture 4: 11th February 2009
The theory of measurement and the origin of randomness

Lecture 5: 18th February 2009
Nonlocality, relativistic spacetime, and quantum equilibrium

Lecture 6: 25th February 2009
Calculating things with quantum trajectories

Lecture 7: 4th March 2009
Not even wrong. Why does nobody like pilot-wave theory?

Lecture 8: 11th March 2009
Bohmian metaphysics : the implicate order and other arcana
Followed by a GENERAL DISCUSSION.

Slides/references on web site: `www.tcm.phy.cam.ac.uk/~mdt26/pilot_waves.html`